Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 27, No. 4, pp. 369–386, 1996

0362-546X/96 $15.00 + 0.00

0362-546X(95)00027-5

EXPONENTIAL DICHOTOMY AND EXISTENCE OF PSEUDO ALMOST-PERIODIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS*

E. AIT DADS† and O. ARINO‡

†Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P.S.15, Marrakech, Morocco; and

‡Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l' Adour, URA 1204, 64000 Pau, France

(*Received* 8 *July* 1994; *received in revised form* 23 *January* 1995; *received for publication* 21 *February* 1995)

Key words and phrases: Almost-periodic, pseudo almost-periodic, exponential-dichotomy, mean value Module, ergodic perturbation, eigenvalues, Floquet multipliers, Lyapounov functions, Fourier exponents, differential equations, asymptotically almost-periodic, stable and unstable spaces, trace.

1. INTRODUCTION

In [1], C. Zhang has introduced an extension of the almost periodic functions, the so-called pseudo almost periodic functions (abbrev. as p.a.p. functions) (for more details on this notion, we can refer to [2]).

A well known extension of almost periodic functions is the class of asymptotically almost periodic functions (which was introduced by Frechet), that is, functions of the type

$$f = g + \epsilon$$

where g is almost-periodic and $\epsilon(t) \to 0$ as $t \to \infty$.

The definition of a p.a.p. given in [1] is as follows: any function f which can be written as a sum

$$f = g + \varphi$$

where g is almost-periodic and φ is continuous, bounded and $M(\|\varphi\|) = 0$. $M(.)$ is the "asymptotic" mean value, defined by

$$M(\psi) = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \psi(s)\, \mathrm{d}s.$$

The main purpose of Zhang's [1] paper is to investigate existence of pseudo almost periodic solutions of a pseudo almost periodic nonlinear perturbation of a linear autonomous ordinary differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + f(t) + \mu G(x(t), t). \qquad (1)$$

*This work is supported by the CNRS program of environment and the Med-Campus program.

In this paper, we first introduce a slight extension of the notion of pseudo almost periodic function, namely, we do not assume that φ is bounded, nor that it is continuous.

We will then consider a nonlinear perturbation of the type (1), assuming that A is time dependant, and has pseudo almost periodic coefficients. Under the assumption that the linear equation has an exponential dichotomy we will prove an analogous existence theorem concerning pseudo almost periodic solutions, using the technique of exponential dichotomy.

Our work is organized as follows: in Section 2, we state some facts on generalized pseudo almost periodic functions; in Section 3, we recall the notion of exponential dichotomy; in section 4, we prove the main theorem on pseudo almost periodic solutions of the linear system; in Section 5, we study a perturbed linear system. Finally, we give a few examples.

2. SOME RESULTS ON GENERALIZED PSEUDO ALMOST PERIODIC FUNCTIONS

Set

$$\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R}) = \left\{ \varphi : \mathbb{R} \to \mathbb{R}, \text{ Lebesgue measurable, such that } \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)| \mathrm{d}s = 0 \right\}$$

and

$$\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\Omega \times \mathbb{R}) = \left\{ \begin{array}{l} \varphi : \Omega \times \mathbb{R} \mapsto \mathbb{R}^n, \text{ such that } \varphi(x,.) \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R}), \text{ for each } \times \in \Omega \text{ and} \\ \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(x,s)\| \mathrm{d}s = 0 \text{ uniformly in } x \in \Omega, \text{ where } \Omega \subset \mathbb{R}^n \end{array} \right\}$$

where Ω is a subset of $\mathbb{R}^n$.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ (respectively $\Omega \times \mathbb{R} \to \mathbb{R}^n$) is called generalized pseudo almost periodic (pseudo almost periodic in $t \in \mathbb{R}$, uniformly in $x \in \Omega$) if

$$f = g + \varphi$$

where $g \in \mathscr{A}\mathscr{P}(\mathbb{R})$ ($\mathscr{A}\mathscr{P}(\Omega \times \mathbb{R})$) and $\varphi \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})$ $\left(\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\Omega \times \mathbb{R})\right)$.

For some preliminary results on such functions we refer to [1].

Remark 1. Note that g and φ are uniquely determined. Indeed, since

$$N(\varphi) = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)| \mathrm{d}s$$

is a norm on $\mathscr{A}\mathscr{P}(\mathbb{R})$, then if $f \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}(\mathbb{R})$, $f = g_1 + \varphi_1 = g_2 + \varphi_2$ one has $N(g_1 - g_2) = 0$, which implies that $g_1 = g_2$, thus, $\varphi_1 = \varphi_2$. g and φ are called the almost periodic component and the ergodic perturbation, respectively, of the function f. Denote by $\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}(\mathbb{R})$;(resp. $\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}(\Omega \times \mathbb{R})$) the set of generalized pseudo almost periodic functions (resp. generalized pseudo almost periodic functions, uniformly in $x \in \Omega$)

Example.

$$f(t) = \sin t + \sin \pi t + \frac{1}{\sqrt{|t|}}$$

$$F(x,t) = f(t) \cos x.$$

2.1. *Properties*

Remark 1. Similarly as in the case of almost periodic functions we can define the mean value, the Fourier coefficients and the Fourier exponents for any generalized pseudo almost periodic function. These are the mean value, the Fourier coefficients and the Fourier exponents of its almost periodic component.

LEMMA 2.2. If $f \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})$, then

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} f(s)\,\mathrm{d}s = M(f)$$

exists and is finite. It is the mean value of f. Moreover

$$M(f) = M(g).$$

Proof of lemma 2.2. Indeed

$$\lim_{r\to\infty} \frac{1}{2} \int_{-r}^{r} f(s)\,\mathrm{d}s = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} g(s)\,\mathrm{d}s + \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \varphi(s)\,\mathrm{d}s$$

Since $g \in \mathscr{A}\mathscr{P}$ then

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} g(s)\,\mathrm{d}s$$

exists and is finite [3]. Furthermore, we have

$$-|\varphi(s)| \leq \varphi(s) \leq |\varphi(s)|$$

Then

$$-\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)|\,\mathrm{d}s \leq \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \varphi(s)\,\mathrm{d}s \leq \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)|\,\mathrm{d}s$$

So

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \varphi(s)\,\mathrm{d}s = 0 = M(\varphi) \qquad \text{and} \qquad M(f) = M(g).$$

It is known [3] that for each $\lambda \in \mathbb{R}$, the function $f(s)\exp(-i\lambda s)$ is also pseudo almost periodic, with the almost periodic component equal to $g(s)\exp(-i\lambda s)$. So, we can define the Fourier coefficients for $f \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}$

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} f(s)\exp(-i\lambda s)\,\mathrm{d}s = a(f, \lambda)$$

exists for all almost periodic functions and any real number λ. Moreover, there exists a countable set of real numbers Λ, such that

$$a(f,\lambda)=0 \qquad \text{if } \lambda \notin \Lambda$$

So if $f=g+\varphi$ is an element of $\tilde{\mathcal{PAP}}(\mathbb{R})$ then $a(f,\lambda)=a(g,\lambda)$.

Remark 2. Pseudo almost periodic functions admit the same Fourier exponents as their almost periodic component. The same is true for the Fourier coefficients. While the Fourier series of a function in $\mathcal{AP}$, this cannot be said for functions in $\tilde{\mathcal{PAP}}$.

LEMMA 2.3. If $f\in\tilde{\mathcal{PAP}}$ $(\mathbb{R})$, and if g is its almost periodic component, then we have

$$g(\mathbb{R})\subset\overline{f(\mathbb{R})}$$

and furthermore if φ is continuous and bounded, we have

$$\|f\|\geq\|g\|\geq \inf_{t\in R}|g(t)|\geq \inf_{t\in R}|f(t)|$$

PROPOSITION 2.4. If $f:\mathbb{R}\to\mathbb{R}$ $(\mathbb{R}\times\Omega\to\mathbb{R})$

$$f=g+\varphi$$

where $g\in\mathcal{AP}(\mathbb{R})(\mathcal{AP}(\Omega\times\mathbb{R}))$ and $\varphi\in\tilde{\mathcal{PAP}}_0(\mathbb{R})$ $(\tilde{\mathcal{PAP}}_0(\Omega\times\mathbb{R}))$, then

i) If $\lim_{|t|\to\infty}\varphi(t)$ exists, we have $\lim_{|t|\to\infty}\varphi(t)=0$.

ii) If $f\geq 0$, then $g\geq 0$.

Proof. i) Suppose that the property does not hold, then there exists a constant $\alpha>0$, and $t_0\in\mathbb{R}$, such that $\varphi(t)>\alpha$ for $t\geq t_0$, which yields

$$\frac{1}{r}\int_0^r|\varphi(s)|\,\mathrm{d}s=\frac{1}{r}\left[\int_0^{t_0}|\varphi(s)|\,\mathrm{d}s+\int_{t_0}^r|\varphi(s)|\,\mathrm{d}s\right]\geq\frac{1}{r}\alpha(r-t_0).$$

Passing to the limit as $r\to\infty$, we obtain

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^r|\varphi(s)|\,\mathrm{d}s\geq\alpha,$$

which contradicts the fact that $\varphi\in\tilde{\mathcal{PAP}}_0(\mathbb{R})$.

ii) For the proof of this part we need the following lemma

LEMMA 2.5. Suppose $g\in\mathcal{AP}$ $(\mathbb{R})$ is such that for every $\epsilon>0$,

$$\frac{\operatorname{meas}\{t:g(t)>-\epsilon,t\in[-r,+r]\}}{2r}\to 1,\qquad \text{as } r\to+\infty.$$

Then, $g\geq 0$

Proof. We will make a proof by contradiction.

Suppose the conclusion does not hold. This implies that $g(x_0)<0$ for some x_0. Choose $\epsilon>0$, $\epsilon<-g(x_0)$.

By continuity, there exists $\delta > 0$ so that: $|x - x_0| \le \delta \Rightarrow g(x) < -\epsilon$. In view of the Bohr characterization of almost periodicity ([4]), there exists $l > 0$, so that in each interval I, of length L, one can find almost period τ with the property that

$$|g(x+\tau) - g(x)| < \frac{\epsilon}{2}.$$

Choose a sequence τ_k of almost periods, $\tau_k \in [kl, (k+1)l[$ we have

$$g(x+\tau_k) < -\frac{\epsilon}{2}, \qquad \text{for } x \in [x_0 - \delta, x_0 + \delta] \text{ and every } k \in N.$$

Denote $M = |x_0| + \delta$.

We have

$$\text{meas}\left\{x \in [-kl - M, kl + M] : g(t) < -\frac{\epsilon}{2},\right\} \ge 2k\delta.$$

So,

$$\frac{\text{meas}\left\{x \in [-kl - M, kl + M] : g(t) < -\frac{\epsilon}{2},\right\}}{2kl + 2M} \ge \frac{2k\delta}{2kl + 2M}.$$

The right hand side does not tend to zero as $k \to +\infty$. This contradicts the assumption made in the lemma. So $g \ge 0$.

Now, we give the proof of the second part of proposition 2.4.

Assuming $f \ge 0$, we want to show that $g \ge 0$. We have $f = g + \varphi$, with

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)| \mathrm{d}s = 0$$

(But, we are not assuming here that $\varphi(t) \to 0$ as $t \to \pm\infty$.)

$$\exists t_n \to +\infty, g(t + t_n) \to g(t), \qquad \forall t$$

$$\forall \epsilon > 0 / \forall r > 0; \qquad \text{meas}\,\{t \in [-r, +r] : \varphi(t) > \epsilon\} \to 0, \qquad \text{as } r \to \infty$$

which implies that

$$\frac{\text{meas}\,\{t : g(t) > -\epsilon, t \in [-r, +r]\}}{2r} \to 1, \quad \text{as } r + \to \infty.$$

This yields that $g(t) \ge 0\ \forall t$. This completes the proof of proposition 2.4.

Remark 3. There exist functions φ such that φ is in $\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})$, but $\lim_{|t| \to \infty} \varphi(t)$ does not exist. Consider, for example,

$$\varphi(t) = \begin{cases} \sqrt{n}, & n \le t \le n + \dfrac{1}{n} \\ 0, & t \text{ elsewhere} \end{cases}$$

It is clear that $\lim_{|t| \to \infty} \varphi(t)$ does not exist

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)| \mathrm{d}s = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\sqrt{k}}{kn} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 0$$

PROPOSITION 2.6. $\tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})$ is a translation invariant C^* −algebra of $C(\mathbb{R})$ containing the constant functions. Furthermore

$$\tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})/\tilde{\mathscr{P}\mathscr{A}\mathscr{P}}_0(\mathbb{R}) = \mathscr{A}\mathscr{P}(\mathbb{R})$$

The proof is similar to the one given in the precedent section.

For $H = (h_1, h_2, .., h_n) \in (C(\mathbb{R}))^n$, suppose that $H(t) \in \Omega$, for all $t \in \mathbb{R}$. Define

$$F : \mathbb{R} \to \Omega \times \mathbb{R}$$

by

$$F(t) = (h_1(t), h_2(t), \ldots, h_n(t), t)$$

For $f = (f_1, f_2, .., f_n) \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})^n$, let $G = (g_1, g_2, .., g_n)$ and $\Phi = (\varphi_1, \varphi_2, .., \varphi_n)$ where g_i and φ_i are the almost periodic component and the ergodic perturbation, respectively of f_i, $i = 1,2,\ldots n$.

THEOREM 2.7. [1] For $i = 1,2,\ldots n$, let $\Omega_i \subset \mathbf{C}$ be a closed set, $\Omega = \Pi_1^n \ \Omega i \subset \mathbf{C}^n$. Let $f \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\Omega \times \mathbb{R})$ satisfy

$$|f(x,t) - f(y,t)| \le L \sum_1^n |x_i - y_i|, \qquad x, y \in \Omega, t \in \mathbb{R},$$

for some $L > 0$. If $H \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})^n$ and $H(t) \in \Omega$, for all $t \in \mathbb{R}$, then f o $F \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}$ $(\Omega \times \mathbb{R})$

3. EXPONENTIAL DICHOTOMY

Let $A(t)$ be a continuous square matrix on an interval J and let $X(t)$ be a fundamental matrix of the system

$$\frac{dx}{dt}(t) = A(t)x(t), \tag{2}$$

satisfying $X(0) = I$, where I is the unit matrix.

A system of differential equations (2) is said to possess an exponential dichotomy on the interval J, if there exists a projection matrix P $(P = P^2)$ and constants $k > 1$, $\alpha > 0$, so that

$$\begin{cases} \|X(t)PX^{-1}(s)\| \le k \exp(-\alpha(t-s)), & \text{for } s \le t, \\ \|X(t)(I-P)X^{-1}(s)\| \le k \exp(-\alpha(s-t)), & \text{for } t \le s. \end{cases} \tag{3}$$

We denote by (P, k, α) a triple of elements associated to an exponential dichotomy.

We use the notation $\mathscr{P}(t)$ for the projection matrix function $X(t)PX^{-1}(t)$. Note, that for all t and s

$$X(t)PX^{-1}(s) = X(t)X^{-1}(s)\mathscr{P}(s) = \mathscr{P}(t)X(t)PX^{-1}(s), \tag{4}$$

$$\begin{aligned} X(t)(I-P)X^{-1}(s) &= X(t)X^{-1}(s)(I - \mathscr{P}(s)) \\ &= (I - \mathscr{P}(t))X(t)PX^{-1}(s), \end{aligned} \tag{5}$$

and that $\mathscr{P}(t)$ is a solution of the matrix system

$$\frac{dX}{dt}(t) = A(t)X - XA(t). \tag{6}$$

So, $\mathscr{P}(t)$ is determined by its value at one point [5]. When $J = [0, \infty[$, the range of $\mathscr{P}(0)$ must be the stable subspace

$$\left\{ \xi \in \mathbb{R}^n ; \sup_{t \geq 0} \| X(t) X^{-1}(0) \xi \| < \infty \right\}, \tag{7}$$

but the kernel may be any complementary subspace. When $J =]-\infty, 0]$, the kernel of $\mathscr{P}(0)$ is the unstable subspace

$$\left\{ \xi \in \mathbb{R}^n ; \sup_{t \leq 0} \| X(t) X^{-1}(0) \xi \| < \infty \right\}. \tag{8}$$

However, the range may be any complementary subspace. So, when $I = \mathbb{R}$, $\mathscr{P}(t)$ is uniquely determined.

In [6] Coppel shows the following result.

PROPOSITION 3.1. Let $A(t)$ be an $n \times n$ matrix function, defined and continuous on $\mathbb{R}$. Then system (2) has an exponential dichotomy on $\mathbb{R}$ if and only if it has an exponential dichotomy on both $[0, \infty[$ and $]-\infty, 0]$, and $\mathbb{R}^n$ is the direct sum of the stable and unstable subspaces.

LEMMA 3.2. Let $A(t)$ be an $n \times n$ matrix function, defined and continuous on $[0, \infty[$ such that the system (2) has an exponential dichotomy (P, k, α) on $[0, \infty[$, $t \to B(t)$ is an application from $\mathbb{R}$, with values in $l(\mathbb{R}^n)$, (the space of linear and continuous applications from $\mathbb{R}^n$ to $\mathbb{R}^n$) continuous and uniformly bounded in t. If

$$\delta = \sup_{t \geq 0} |B(t)| < \frac{\alpha}{4k^2} \tag{9}$$

Then, the system

$$\frac{dx}{dt} = [A(t) + B(t)]x \tag{10}$$

has an exponential dichotomy $(Q,\ 5/2k^2,\ \alpha - 2k\delta)$ where Q is a matrix projection and $(5/2k^2,\ \alpha - 2k\delta)$ the constants of dichotomy.

COROLLARY 3.3. Let $A(t)$ be an $n \times n$ matrix function, defined and continuous on $[0, \infty[$, such that the system (2) has an exponential dichotomy (P, k, α) on $[0, \infty[$. Let $t \to B(t)$ be another matrix function continuous and such that

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |B(s)| ds = 0.$$

Then, the perturbed system (10) also has an exponential dichotomy, and if $\mathscr{Q}(t)$ is the corresponding projection matrix function, then

$$|\mathscr{P}(t) - \mathscr{Q}(t)| \to 0, \qquad \text{as } t \to \infty.$$

Proof. Let k, α, be the constants involved in the exponential dichotomy for (2). From proposition 2.4, $B(t) \to 0$, as $t \to \infty$. So, there exists t_0 such that $|B(t)| < \alpha/4k^2$ for $t \geq t_0$ for any $\sigma \geq t_0$ by applying lemma 3.2 it follows that (10) has an exponential dichotomy on $[\sigma, \infty[$ with constants k_1, α_1, depending only on k, α, and the associated matrix function $\mathscr{Q}_\sigma(t)$ can be so chosen so that for $t \geq \sigma$

$$|\mathscr{P}(t) - \mathscr{D}_\sigma(t)| \leq 4\alpha^{-1}k^3\delta(\sigma),$$

where

$$\delta(\sigma) = \sup_{t \geq \sigma}|B(t)|.$$

It follows that the equation (10) has an exponential dichotomy on $[t_0, \infty[$, since it has one on $[\sigma, \infty[$ [6]. Let $\mathscr{D}(t)$ be any allowable projection matrix function with associated constants L, β. $\mathscr{D}(t)$ and $\mathscr{D}_\sigma(t)$ must have the same range for all t so that

$$\mathscr{D}(t)\mathscr{D}_\sigma(t) = \mathscr{D}_\sigma(t), \qquad \mathscr{D}_\sigma(t)\mathscr{D}(t) = \mathscr{D}(t).$$

Then, if $Y_\sigma(t)$ is the fundamental matrix for (10), with $Y_\sigma(\sigma) = I$,

$$|\mathscr{D}_\sigma(t) - \mathscr{D}(t)| = |\mathscr{D}_\sigma(t)(I - \mathscr{D}(t))| = |Y_\sigma(t)\mathscr{D}_\sigma(\sigma)(I - \mathscr{D}(\sigma))Y_\sigma^{-1}(t)|$$
$$\leq |Y_\sigma(t)\mathscr{D}_\sigma(\sigma)||(I - \mathscr{D}(\sigma))Y_\sigma^{-1}(t)| \leq k_1 \exp(-\alpha_1(t-\sigma))L \exp(-\beta(t-\sigma)),$$

for $t \geq \sigma \geq t_0$ (here we have used (3)). So, if $\sigma \geq t_0$ and $t \geq 2\sigma$

$$|\mathscr{P}(t) - \mathscr{D}(t)| \leq |\mathscr{D}_\sigma(t) - \mathscr{D}(t)| + |D_\sigma(t) - \mathscr{P}(t)|$$
$$\leq k_1 L \exp(-(\alpha_1 + \beta)\sigma)) + 4\alpha^{-1}k^3\delta(2\sigma).$$

This last quantity tends to zero as $\sigma \to \infty$, it follows that $|\mathscr{P}(t) - \mathscr{D}(t)| \to 0$, as $t \to \infty$.

COROLLARY 3.4. If $A(t)$ is a pseudo almost periodic $n \times n$ matrix such that the almost periodic part of system (2) has an exponential dichotomy, then (10) also has an exponential dichotomy.

When $A(t) = A$ is constant, (2) has an exponential dichotomy on an infinite interval if and only if the eigenvalues of A have a nonzero real part. When $A(t)$ is periodic, (2) has an exponential dichotomy on an infinite interval if and only if the Floquet multipliers lie off the unit circle. The dimensions of the stable subspaces are the number of eigenvalues with negative real parts (or the number of multipliers inside the unit circle).

4. THE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

For the column vector

$$G = (g_1, g_2, \ldots \ldots g_n)^T \in \mathscr{C}(\mathbb{R})^n,$$

define

$$\|G\| = \max\{\|g_i\| : 1 \leq i \leq n\},$$

where

$$\|g_i\| = \sup_{t \in R}|g_i(t)|,$$

for any bounded function.

In this section, we consider the following system

$$\frac{dx}{dt}(t) = A(t)x(t) + F(t), \tag{11}$$

where $A(t) = (a_{ij}(t))$ is an $n \times n$ almost periodic matrix and $F(t) = (f_1, f_2, \ldots, f_n)^T \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}(\mathbb{R})^n$.

THEOREM 4.1. Let $A(t)=(a_{ij}(t))$ be an $n\times n$ almost periodic matrix. Suppose that the system $(\mathrm{d}x/\mathrm{d}t)(t)=A(t)x(t)$ satisfies an exponential dichotomy and suppose that $F\in\mathscr{PAP}(\mathbb{R})^n$. Then (11), has a generalized pseudo almost periodic solution. Furthermore if $F\in\mathscr{PAP}(\mathbb{R})^n$, then the pseudo almost periodic solution is unique, and one has $\|x\|\leq(2k/\alpha)\|F\|$

$$\mathrm{mod}(x)\subset\mathrm{mod}\,A+\mathrm{mod}F.$$

For the proof of theorem 4.1, we need the following results

LEMMA 4.2. If $A(t)$ is almost periodic and determines an exponential dichotomy, then the only solution with at most polynomial growth both at $+\infty$ and $-\infty$, of the homogeneous system

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=A(t)x(t) \tag{12}$$

is $x=0$.

Proof. In the case, where A determines an exponential dichotomy, there exists a Lyapunov function $V(x,t)$ such that (see [7])

$$-c\|x\|^2\leq V(x,t)\leq c\|x\|^2 \qquad \text{and} \qquad \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_{(2)}(x,t)\geq k\|x\|^2 \tag{13}$$

$(\mathrm{d}V/\mathrm{d}t)_{(2)}$ (x,t) denotes the derivative of V with respect to the equation (2) We have two cases:

Case 1. Suppose now that

$$V(x(t),t)\leq 0, \qquad \forall t\in\mathbb{R},$$

let us denote $W(t)=V(x(.),.)(-t)$. Then, one has

$$\frac{\mathrm{d}W}{\mathrm{d}t}=-\frac{\mathrm{d}}{\mathrm{d}t}[V(x)(.),.)(-t)]\leq -k\|x\|^2\leq\frac{k}{c}W(t).$$

From (13), we can see that

$$\frac{\mathrm{d}V(x(t),t)}{\mathrm{d}t}>0,$$

$W(t)$ is nonincreasing, the $W(t)<W(0)$, which given the following

$$\frac{\mathrm{d}W}{\mathrm{d}t}\leq\frac{k}{c}W(0) \qquad \text{for } t\geq 0$$

Since $W(t)<0$, there exists $k_0>0$, such that

$$\frac{\mathrm{d}W}{\mathrm{d}t}\leq -k_0. \qquad \text{So, } W(t)\leq -k_0t, \qquad \text{for } t\geq 0.$$

Inductively, we get

$$W(t)\leq\frac{k_0^n}{n!}t^n, \qquad \text{for } t\geq 0$$

Then, by using inequality (13) we have that

$$\|x\| \geq c\frac{k_0^n}{n!}t^n, \qquad \text{for all } t \geq 0, \qquad \text{and} \qquad n \in \mathcal{N}$$

So, x grows faster than any polynomial at $+\infty$.

Case 2. V changes signs. In this case, there exists $t_0 \in \mathbb{R}$, such that

$$V(x(t), t) > 0, \qquad \text{for all } t \geq t_0,$$

Using the formula (13), we have the following inequality.

$$W(t) \leq W(0)\exp\frac{k}{c}t, \quad \text{for } t \geq t_0$$

which leads to the existence of constants $\beta, \alpha > 0$, such that

$$x(t) \geq \beta \exp \alpha t$$

The proof of lemma is complete. ■

Proof of theorem 4.1. We will first show uniqueness. For this purpose, we will use the lemma 4.2, that is we will show that every generalized pseudo almost periodic solution of 12 grows at most polynomially. So, taking two solutions x_1, x_2. The difference grows at most polynomially and is a solution of equation (12). So, by lemma 4.2, the difference $x_1 - x_2 = 0$. So, let x be a generalized pseudo almost periodic solution, let us prove that x is polynomially bounded. If x is a generalized pseudo almost periodic solution of (11) then, we can write $x(t) = y(t) + z(t)$, where y is the almost periodic component. If x is pseudo almost periodic and $\mathrm{d}x/\mathrm{d}t$ is pseudo almost periodic, let us put

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(t) + v(t), \qquad \text{with } u \text{ almost-periodic.}$$

Then

$$x(t) = \int_0^t u(s)\,\mathrm{d}s + \int_0^t v(s)\,\mathrm{d}s + x(0)$$

We know that $\int_0^t u(s)\mathrm{d}s$ is almost periodic if and only if $\int_0^t u(s)\,\mathrm{d}s$ is bounded [4].

Since u is almost periodic, then u is bounded, so, there exists M such that $|u(t)| \leq M$, furthermore, $v \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})^n$ which yields that, there exists M_1 such that $\int_0^t v(s)\mathrm{d}s| \leq M_1 t$. Consequently $|x(t)| \leq (M + M_1)t$, which is the desired result.

Now, if we consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t)x(t) + F(t),$$

and x is a generalized pseudo almost periodic solution with F generalized pseudo almost periodic, then $A(t)x(t) + F(t)$ is a generalized pseudo almost periodic, so $(\mathrm{d}x/\mathrm{d}t)$ (t) is pseudo almost periodic, which implies that $x(t)$ is $o(|t|)$ as $\pm\infty$. However, the equation 12 has at most one solution with at most polynomial growth, then, (the unique bounded solution, when we consider F bounded) a solution $x(t)$ is represented as follows (see [8])

$$x(t) = \int_{-\infty}^{\infty} G(t, s)F(s)\,\mathrm{d}s, \tag{14}$$

where

$$G(t,s)=\begin{cases} X(t)PX^{-1}(s), & \text{for } t\geq s,\\ X(t)(I-P)X^{-1}(s), & \text{for } t\leq s.\end{cases} \tag{15}$$

$G(t,s)$ is a piecewise continuous function on the (t,s) plane, is a Green function

$$\|x\|=\|\int_{-\infty}^{\infty}G(t,s)F(s)\,\mathrm{d}s\|\leq\int_{-\infty}^{\infty}\|G(t,s)F(s)\|\,\mathrm{d}s\leq\int_{-\infty}^{\infty}\|G(t,s)\|\,\|F(s)\|\,\mathrm{d}s$$

$$\leq\|F\|\int_{-\infty}^{\infty}\|G(t,s)\|\,\mathrm{d}s\leq\|F\|\int_{-\infty}^{t}\|G(t,s)\|\,\mathrm{d}s+\|F\|\int_{t}^{\infty}\|G(t,s)\|\,\mathrm{d}s$$

(using the property of exponential dichotomy (3))

$$\|x\|\leq\|F\|\int_{-\infty}^{t}\|X(t)PX^{-1}(s)\|\,\mathrm{d}s+\|F\|\int_{t}^{\infty}\|X(t)(I-P)X^{-1}(s)\|\,\mathrm{d}s$$

$$\leq\|F\|\int_{-\infty}^{t}k\exp\left(-\alpha(t-s)\right)\mathrm{d}s+\|F\|\int_{t}^{\infty}k\exp(-\alpha(s-t))\,\mathrm{d}s$$

$$\leq\|F\|\frac{k}{\alpha}\left[\exp(-\alpha(t-s)|_{-\infty}^{t}+\exp(-\alpha(s-t)|_{\infty}^{t}\right]$$

$$\leq\|F\|\frac{k}{\alpha}(1+1)=2\frac{k}{\alpha}\|F\|.$$

This gives the second part of theorem 4.1. Now, if

$$F(t)=g(t)+\varphi(t),\qquad\text{where } g\in\mathscr{A}\mathscr{P}(\mathbb{R})^n,\qquad\text{and}\qquad\varphi\in\tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})^n$$

Then

$$x(t)=\int_{-\infty}^{\infty}G(t,s)g(s)\,\mathrm{d}s+\int_{-\infty}^{\infty}G(t,s)\varphi(s)\,\mathrm{d}s.$$

It is known [9] that

$$\int_{-\infty}^{\infty}G(t,s)g(s)\,\mathrm{d}s$$

is an almost periodic function. If we put

$$\Phi(t)=\int_{-\infty}^{\infty}G(t,s)\varphi(s)\,\mathrm{d}s,$$

in order to show that x is in $\tilde{\mathscr{P}\mathscr{A}\mathscr{P}}(\mathbb{R})^n$, we need to show that $\Phi \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}_0(\mathbb{R})^n$, i.e., we need to show that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-\infty}^{r}\|\Phi(t)\|\,dt=0. \tag{16}$$

$$0\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\Phi(t)\|\,dt=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left\|\int_{-\infty}^{\infty}G(t,s)\varphi(s)\,ds\right\|dt$$

$$\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\int_{-\infty}^{\infty}\|G(t,s)\varphi(s)\|\,ds$$

$$\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\int_{-\infty}^{\infty}\|G(t,s)\varphi(s)\|\,ds$$

$$\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-\infty}^{t}\|G(t,s)\varphi(s)\|\,ds+\int_{t}^{\infty}\|G(t,s)\varphi(s)\|\,ds\right)$$

$$\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-\infty}^{t}\|X(t)PX^{-1}(s)\varphi(s)\|\,ds\right.$$

$$\left.+\int_{t}^{\infty}\|X(t)(I-P)X^{-1}(s)\varphi(s)\|\,ds\right)$$

$$=I_1+I_2$$

Now

$$0\le I_1\le\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-\infty}^{t}k\exp(-\alpha(t-s))\,\|\varphi(s)\|\,ds\right.$$

$$=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-\infty}^{-r}k\exp(-\alpha(t-s))\|\varphi(s)\|\,ds\right.$$

$$\left.+\int_{-r}^{t}k\exp(-\alpha(t-s))\|\varphi(s)\|\right)ds$$

$$=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-r}^{t}k\exp(-\alpha(t-s))\,\|\varphi(s)\|\,ds\right)$$

$$+\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}dt\left(\int_{-\infty}^{-r}k\exp(-\alpha(t-s))\,\|\varphi(s)\|\,ds.\right.$$

$$=I_1^1+I_1^2$$

Now,

$$I_1^1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \int_{-r}^{t} k \exp(-\alpha(t-s)) \mathrm{d}s$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \left(\frac{k}{\alpha} \exp(-\alpha(t-s))|_{-r}^{t} \right)$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \left(\frac{k}{\alpha} (1 - \exp(-\alpha(t+r)) \right)$$

Since

$$-r \le t \le r \text{ and } \alpha > 0, \text{ then } k/\alpha\ (1-\exp(-\alpha(t+r))$$

is bounded. Furthermore, $\varphi \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})^n$, the $I_1^1 \to 0$, as $r \to +\infty$.

Also

$$I_1^2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t \int_{-\infty}^{-r} \|\varphi(s)\| k \exp(-\alpha(t-s)) \mathrm{d}s,$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \mathrm{d}t \int_{-r}^{r} k \exp(-\alpha(t-s)) \mathrm{d}s,$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \mathrm{d}t \left(\frac{k}{\alpha} [\exp(-\alpha(t-r)) - \exp(-\alpha(t+r))] \right),$$

$$= \frac{k}{\alpha} \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-\alpha(t-r)) \mathrm{d}t$$

$$- \frac{k}{\alpha} \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-\alpha(t+r)) \mathrm{d}t.$$

If we consider the quantity

$$h(r) = \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-\alpha(t-r)) \mathrm{d}t,$$

and, integrate by parts, we obtain

$$h(r) = \frac{1}{2r} \left[\left(\int_0^t \|\varphi(u)\| \mathrm{d}u \right) \exp(-\alpha(t-r)) \right]_r^{+\infty}$$

$$+ \frac{\alpha}{2r} \int_r^{\infty} \left(\int_0^t \|\varphi(u)\| \mathrm{d}u \right) \exp(-\alpha(t-r)) \mathrm{d}t.$$

It is clear that the first term in the above equality tends to zero as $r \to \infty$ and, by hypothesis $\varphi \in \tilde{\mathscr{P}}\mathscr{A}\mathscr{P}_0(\mathbb{R})^n$, we have

$$\frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \to 0 \qquad \text{as } r \to \infty.$$

So, we have that there exists $\epsilon(r)$ such that

$$\int_0^t \|\varphi(u)\| \, du \leq \epsilon(r)t.$$

Now, one has

$$\frac{\alpha}{2r}\int_r^\infty\left(\int_0^t \|\varphi(u)\|\,du\right)\exp(-\alpha(t-r))\,dt$$

$$\leq \frac{\epsilon(r)\alpha}{r}\int_r^\infty t\exp(-\alpha(t-r))\,dt$$

$$=\left[\frac{\epsilon(r)\alpha}{r}\frac{-1}{\alpha}t\exp(-\alpha(t-r))\right]_r^{+\infty}-\frac{\epsilon(r)\alpha}{r}\int_r^\infty \exp(-\alpha(t-r))\,dt$$

$$=\frac{\epsilon(r)\alpha}{r}\frac{+1}{\alpha}r-\frac{1}{\alpha}\frac{\epsilon(r)\alpha}{r}=\epsilon(r)\left(1-\frac{1}{r}\right).$$

This last quantity goes to zero as r goes to infinity. By the same arguments, we prove that the quantity

$$-\frac{k}{\alpha}\frac{1}{2r}\int_r^\infty \|\varphi(t)\|\exp(-\alpha(t+r))\,dt$$

goes to zero as r goes to infinity.

Consequently, we have that $I_1^2 \to 0$ as $r\to\infty$.

Consider now the quantity

$$I_2=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^r dt\int_t^\infty \|X(t)(I-P)X^{-1}(s)\varphi(s)\|\,ds.$$

We have

$$I_2\leq \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^r dt\left(\int_t^r k\exp(-\alpha(s-t))\|\varphi(s)\|\,ds\right.$$

$$\left.+\int_r^\infty k\exp(-\alpha(s-t))\|\varphi(s)\|\,ds\right),$$

$$I_2\leq \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^r \|\varphi(t)\|\,dt\left(\int_t^r k\exp(-\alpha(s-t))\,ds\right)$$

$$+\lim_{r\to\infty}\frac{1}{2r}\int_r^\infty\|\varphi(t)\|\,dt\int_{-r}^r k\exp(-\alpha(s-t))\,ds),$$

Let $I_2=I_2^1+I_2^2$

Now,

$$I_2^1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \left(\int_{t}^{r} k \exp(-\alpha(s-t)) \mathrm{d}s \right),$$

$$I_2^1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \left(-\frac{k}{\alpha} \exp(-\alpha(s-t))|_t^r \right),$$

$$I_2^1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t \left(\frac{k}{\alpha}(1 - \exp(-\alpha(r-t))) \right).$$

Since $-r \le t \le r, \alpha > 0$. So $I_2^1 \to 0$ as $r \to \infty$,

$$I_2^2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t \left(\int_{r}^{\infty} k \exp(-\alpha(s-t)) \|\varphi(s)\| \mathrm{d}s \right).$$

$$I_2^2 \le \lim_{r\to\infty} \frac{1}{2r} \int_{-4}^{r} \|\varphi\| \mathrm{d}t \left(-\frac{k}{\alpha} \exp(-\alpha(s-t))|_r^\infty \right),$$

$$I_2^2 \le \frac{k}{\sigma} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| (\exp(-\alpha(r-t)) \mathrm{d}t.$$

Since $-r \le t \le r, \alpha > 0$, then $\exp(-\alpha(r-t)) < 1$. Furthermore, $\varphi \in \tilde{\mathscr{P}\mathscr{A}\mathscr{P}}_0(\mathbb{R})^n$, then $I_2^2 \to 0$, as $r \to +\infty$.

For the last part of the theorem we can refer to Fink [10]. This completes the proof of theorem 4.1 ■

Corollary 4.2. If $A(t)$ is $\bar{\omega}$-periodic with the Floquet multipliers lying off the unit circle. Or, if $A(t) = A$ is a constant matrix, assume that the eigenvalues of A have nonzero real parts, then we have the same conclusion as in theorem 4.1.

5. CASE OF PERTURBED LINEAR SYSTEM

In the following section we consider the perturbed linear system.

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t)x(t) + F(t) + \mu G(x(t), t) \tag{17}$$

$\mu \in \mathbf{C} - \{0\}$ $G \in \mathscr{P}\mathscr{A}\mathscr{P}(\Omega \times \mathbb{R})^n$.

Using the same method as in [1], we have the following theorem

Theorem 5.1. Let A, F be as in theorem 4.1. Let x^0 the unique solution in $\mathscr{P}\mathscr{A}\mathscr{P}(\mathbb{R})^n$ of the generating system of (17). Let $a_i > \|x^0\|$, $i = 1, \dots, n$ and $\boldsymbol{\Omega} = \{z \in \mathbf{C}^n, |z_i| \le a_i\}$, and assume that

1) $G \in \mathcal{PAP}(\mathbb{R})^n$ and such that

$$\|G(x,.) - G(y,.)\| \leq L \sum_{i=1}^{n} |x_i - y_i| \quad x, y \in \boldsymbol{\Omega};$$

2)
$$0 < |\mu| < \min_{1 \leq i \leq n} \left\{ \frac{1}{2nL\frac{k}{\alpha}}, \frac{(a_i - \|x^0\|)}{\frac{2k\|G\|}{\alpha}} \right\},$$

where k, α are exponential dichotomy constants. Then there exists a unique solution $X = (x_1, x_2,.,.,.,x_n)^T \in \mathcal{PAP}(\mathbb{R})^n$ of equation (17) such that $X(t) \in \boldsymbol{\Omega}$ for all $t \in \mathbb{R}$.

Proof. We construct a sequence of approximations by induction. Let $(x^k)_{k \in N}$ be the sequence defined by x^0 is the unique pseudo almost-periodic solution of the equation

$$\frac{dx}{dt}(t) = A(t)x(t) + F(t)$$

and x^k is the unique bounded solution of the equation

$$\frac{dx^k}{dt}(t) = A(t)x^k(t) + F(t) + \mu G(t, x^{k-1}(t))$$

If we put

$$H(t) = F(t) + \mu G(t, x^{k-1}(t))$$

It is easy to see that $H(t) \in \mathcal{PAP}(\mathbb{R})^n$, and that there exists one and only one pseudo almost-periodic solution x^k for this last equation which satisfies

$$\frac{d[x^k - x^0]}{dt}(t) = A(t)[x^k - x^0](t) + \mu G(x^{k-1}(t), t)$$

So

$$\|x^k\| \leq \|x^k - x^0\| + \|x^0\|$$

$$\|x^k\| \leq \frac{2k}{\alpha}\mu\|G\| + \|x^0\|$$

Furthermore, $x^k(\mathbb{R}) \subset \Omega$ since

$$|\mu| < \min_{1 \leq i \leq n} \left\{ \frac{1}{2nL\frac{k}{\alpha}}, \frac{(a_i - \|x^0\|)}{\frac{2k\|G\|}{\alpha}} \right\}$$

Now, using result of the precedent section, one can prove that the sequence $(x^k)_{k \in N}$ is a Cauchy sequence, which is uniformly convergent, with the uniform limit is an element of $\mathcal{PAP}(\mathbb{R})^n$, and this limit is a solution of equation (17)

Example. (Markus and Yamabe [8])

$$\frac{dx}{dt}(t) = A(t)x(t) + F(t) \tag{18}$$

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & -1 + \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix} \tag{19}$$

$$F(t) = \begin{pmatrix} \cos \pi t + \dfrac{1}{1+t^2} \\ \sin \pi t + \dfrac{1}{1+t^2} \end{pmatrix}$$

$F(t)$ is periodic with period $2, A(t)$ is π-periodic and the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$, of $A(t)$ are

$$\lambda_1(t) = \frac{[-1 + i\sqrt{7}]}{4}, \qquad \lambda_2(t) = \frac{[-i - i\sqrt{7}]}{4},$$

and, in particular, the real parts of the eigenvalues are negative. On the other hand, one can verify directly that the vector

$$(-\cos t, \sin t)\exp\left(\frac{t}{2}\right)$$

is a solution of (18).

If $\rho_j = \exp(\lambda_j \omega)$, $j = 1,2,..,n$, are the characteristic multipliers of $A(t)$ where ω is the minimal period, λ_j the eigenvalues of $A(t)$, then, we have

$$\prod_{j=1}^{n} \rho_j = \exp \int_0^{\omega} \operatorname{trace} A(s)\,ds,$$

$$\sum_{j=1}^{n} \lambda_j = \frac{1}{\omega}\int_0^{\omega} \operatorname{trace}(A(s))\,ds \left(\operatorname{mod} \frac{2\pi i}{\omega}\right)$$

One of the characteristic multipliers is exp $(\pi/2)$. The other multiplier is $\exp(-\pi)$ since the product of the multipliers is exp $(-\pi/2)$. So, the system

$$\frac{dx}{dt}(t) = A(t)x(t),$$

has an exponential dichotomy. $A(t)$ is π periodic, then it is almost periodic.

$$\begin{pmatrix} \cos \pi t + \dfrac{1}{1+t^2} \\ \sin \pi t + \dfrac{1}{1+t^2} \end{pmatrix}$$

is a sum of periodic function and an ergodic function because

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left|\left(\frac{1}{1+t^2},\frac{1}{1+t^2}\right)\right|\mathrm{d}t=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\frac{2}{1+t^2}\,\mathrm{d}t$$

$$=\lim_{r\to\infty}\frac{2}{2r}(\arctan(t)|_{-r}^{r})=\lim_{r\to\infty}\frac{2}{r}(\arctan(2r))=0$$

So, system (18) has one and only one pseudo almost periodic solution $x(t)$ with

$$mod(x)\subset\mathbf{Z}+\pi\mathbf{Z}.$$

REFERENCES

[1] ZHANG C., Pseudo almost-periodic solutions of some differential equations, *J. Math. Analysis Applic.* **181**, 62–76 (1994).
[2] AIT DADS E. & EZZINBI K., Positive pseudo almost periodic solution for some non linear delay integral equations (submitted to *J. Cybern*).
[3] AMERIO- L. & PROUSE G., *Almost-Periodic Functions and Functional Equations*. Van Nostrand, New York (1971).
[4] CORDUNEANU C., *Almost-Periodic Functions*. 2nd edn, Chelsea, New York (1989).
[5] ARINO O. & GHABBAR Y., Existence de fronts d'ondes forcés par la méthode de dichotomie exponentielle. Publications internes de Pau, n^0 90/31.
[6] COPPELL W. A., Dichotomy and reducibility, *J. diff. Eqns* **3**, 500–521 (1967).
[7] ZHENSHENG LIN, Theory of linear systems, *Ann. Diff. Eqns*, **6** (2) (1990).
[8] PLAAT O., *Ordinary Differential Equations*. Holden-Day, Inc. (1971).
[9] KURIHARA M., *Quasi periodic Solutions of Quasi periodic Differential Difference Equations* (preprint 1984).
[10] FINK A. M., Almost-Periodic Differential Equations. *Lecture Notes in Mathematics*, Vol. 377, Springer Verlag, New York, Berlin (1974).
[11] ZAIDMAN S., Solutions presque périodiques des équations differentielles abstraites, *Enseign. Math.* **24**, 87–110 (1978).