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PSEUDO ALMOST PERIODIC SOLUTIONS FOR SOME DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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1. INTRODUCTION

A very important question in many practical situations is to know if there exists a mean value of bounded solution x of differential equation

$$\frac{dx}{dt}(t) = f(t, x(t)).$$

By the mean value we understand the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

if it exists. Several authors have found a class of functions which have the mean value—the almost periodic. In [1], Zhang introduced an extension of the almost periodic functions, the so-called pseudo almost periodic functions (p.a.p. functions) (for more details on this notion, see [2–4]).

A well-known extension of almost periodic functions is the class of asymptotically almost periodic functions (which was introduced by Fréchet), that is, functions of the type

$$f = g + \varepsilon,$$

where g is almost periodic, $\varepsilon(t)$ is continuous and $\varepsilon(t) \to 0$ as $t \to \infty$.

The definition of a p.a.p. function given in [1] is as follows: any function f which can be written as a sum

$$f = g + \varphi,$$

where g is almost periodic and φ is continuous, bounded and $M(\|\varphi\|) = 0$. Here $M(\cdot)$ is the "asymptotic" mean value, defined by

$$M(\psi) = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \psi(s)\, ds.$$

The main purpose of Zhang's [1] paper is to investigate existence of pseudo almost periodic solutions of a pseudo almost periodic nonlinear perturbation of a linear autonomous ordinary differential equation

$$\frac{dx}{dt}(t) = Ax(t) + f(t) + \mu G(x(t), t). \tag{1}$$

Recently, Ait Dads and Arino [3] considered the extension of Zhang's situation, by introducing a slight extension of the notion of pseudo almost periodic function, namely, they did not assume that φ is continuous, nor bounded, just the zero mean value condition and then considered a nonlinear perturbation of the type (1), assuming that A is time dependent, with pseudo almost periodic coefficients. Under the assumption that the linear equation has an exponential dichotomy they proved an analogous existence theorem concerning pseudo almost periodic solutions, using the technique of exponential dichotomy. As for the space of almost periodic functions and some of its generalizations, pseudo almost periodic functions have many applications in the theory of differential equations. In this paper we are concerned with the same problem in a general Banach space.

Our work is organized as follows. In Section 2, we state some facts on pseudo almost periodic functions with values in a Banach space; in Section 3, we prove the main theorem on pseudo almost periodic solutions of the linear system; in Section 4, we consider the nonlinear case. Finally, we give a few examples.

2. PRELIMINARIES AND BASIC RESULTS

2.1. Semi inner product

Definition 2.1 (Deimling [5]). Let $(E, \|\cdot\|)$ be a Banach space. The upper semi inner product is defined by

$$\langle x, y\rangle_+ = \|y\| \lim_{t\to 0^+} \frac{1}{t}(\|y + tx\| - \|y\|)$$

and the lower semi inner product by

$$\langle x, y\rangle_- = \|y\| \lim_{t\to 0^+} \frac{1}{t}(\|y\| - \|y - tx\|).$$

Both limits exists for every norm, and they coincide with the inner product, if E is a Hilbert space.

In the case when E is a uniformly convex space we have $\langle x, y\rangle_+ = \langle x, y\rangle_-$.

Definition 2.2 (Hanebaly [6]). $F\colon E \to E$ is called dissipative if there exists a constant $c > 0$, such that

$$\langle F(x) - F(y), x - y\rangle_- \le -c\|x - y\|^2. \tag{2}$$

Remark 2.1. Notice that inequality (2) is essentially weaker than the classical Lipschitz condition

$$\|F(x) - F(y)\| \le k\|x - y\|,$$

but it only guarantees existence of solutions to the right.

2.2. Some results on generalized pseudo almost periodic functions

Definition 2.3 (Bohr [7]). Let $(E, \|\cdot\|)$ be a Banach space. Then

$$f: \mathbb{R} \to E$$

is called almost periodic if

(i) f is continuous, and

(ii) for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t + \tau) - f(t)\| < \varepsilon, \qquad \text{for all } t \in \mathbb{R}.$$

Set

$$\tilde{\mathcal{P}}\mathcal{A}\mathcal{P}_0(\mathbb{R}, E) = \left\{ \begin{array}{c} \varphi: \mathbb{R} \to E, \text{ Lebesgue measurable, such that} \\ \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(s)\| \, ds = 0 \end{array} \right\}$$

and

$$\tilde{\mathcal{P}}\mathcal{A}\mathcal{P}_0(\Omega \times \mathbb{R}, E) = \left\{ \begin{array}{l} \varphi: \Omega \times \mathbb{R} \mapsto E, \text{ such that } \varphi(x, \cdot) \in \tilde{\mathcal{P}}\mathcal{A}\mathcal{P}_0(\mathbb{R}) \text{ for every } x \in \Omega, \text{ and} \\ \qquad \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(x, s)\| \, ds = 0 \\ \text{uniformly in } x \in \Omega, \text{ where } \Omega \subset E. \end{array} \right\}$$

Definition 2.4. A function $f: \mathbb{R} \to E$ is called generalized pseudo almost periodic (generalized pseudo almost periodic in $t \in \mathbb{R}$, uniformly in $x \in \Omega$) if

$$f = g + \varphi$$

where $g \in \mathcal{A}\mathcal{P}(\mathbb{R}, E) \cdot (\mathcal{A}\mathcal{P}(\Omega \times \mathbb{R}, E))$ and $\varphi \in \tilde{\mathcal{P}}\mathcal{A}\mathcal{P}_0(\mathbb{R}, E) \cdot (\tilde{\mathcal{P}}\mathcal{A}\mathcal{P}_0(\Omega \times \mathbb{R}, E)$.

Remark 2.2. Note that g and φ are uniquely determined in terms of f. Indeed, since

$$N(\varphi) = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(s)\| \, ds$$

is a norm on $\mathcal{A}\mathcal{P}(\mathbb{R}, E)$, then, if $f \in \tilde{\mathcal{P}}\mathcal{A}\mathcal{P}(\mathbb{R}, E)$, and $f = g_1 + \varphi_1 = g_2 + \varphi_2$, one has $N(g_1 - g_2) = 0$, which implies that $g_1 = g_2$, and thus, $\varphi_1 = \varphi_2$. Then g and φ are called the almost periodic component and the ergodic perturbation, respectively, of the function f. Denote by $\tilde{\mathcal{P}}\mathcal{A}\mathcal{P}(\mathbb{R}, E)$ the set of generalized pseudo almost periodic functions.

Example. Let

$$f(t) = \sin t + \sin \pi t + t|\sin \pi t|^{t^N} \qquad \text{for } N > 6$$

$$f(x, t) = \cos x(\sin t + \sin \pi t + t|\sin \pi t|^{t^N}) \qquad \text{for } N > 6.$$

Let us prove that $\varphi(t) = t|\sin \pi t|^{t^N}$, has zero mean value, while being unbounded.

$\varphi(t) \to \infty$ at the points $t = \frac{1}{2} + k$, as $|k| \to \infty$.

Let us prove that

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r \|\varphi(s)\|\,\mathrm{d}s = 0.$$

Let us look at the integral

$$\int_k^{k+1} \varphi(t)\,\mathrm{d}t = \int_k^{k+1} t|\sin \pi t|^{t^N}\,\mathrm{d}t \le 2(k+1)\int_0^{1/2} |\sin \pi t|^{(k+t)^N}\,\mathrm{d}t$$

$$\le 2(k+1)\int_0^{1/2} |\sin \pi t|^{k^N}\,\mathrm{d}t.$$

We will show that

$$\int_k^{k+1} \varphi(t)\,\mathrm{d}t$$

is bounded above by the general term of a convergent series, which will complete the proof of the example.

We know that

$$\pi\int_0^a (\sin \pi t)^A \cos \pi t\,\mathrm{d}t = \frac{1}{A+1}(\sin \pi a)^{A+1} \qquad \text{with } a < \frac{1}{2}$$

which implies that

$$\int_0^a (\sin \pi t)^A\,\mathrm{d}t \le \frac{1}{\pi(A+1)}\frac{(\sin \pi a)^{A+1}}{\cos \pi a}.$$

On the other hand, we have

$$\frac{(\sin \pi a)^{A+1}}{\cos \pi a} = \frac{\exp[(A+1)/2]\log \sin^2 \pi a}{\cos \pi a} \le \frac{\exp[(A+1)/2](\sin^2 \pi a - 1)}{\cos \pi a}.$$

Now, let us study the function

$$\frac{\exp(-[(A+1)/2]u^2)}{u} \qquad u = \cos \pi a.$$

If we put $B = (A+1)/2$ (which condition guarantees the fact that $(\exp(-Bu^2))/u < 1$).

Consider

$$\varphi(v) = \frac{\exp(-v^2)}{v},$$

then

$$\frac{\exp(-Bu^2)}{u} = \varphi(\sqrt{B}u)\sqrt{B}.$$

$$\frac{\exp(-v^2)}{v} < 1 \Leftrightarrow \varphi(\sqrt{B}u) < \frac{1}{\sqrt{B}}.$$

Since φ is nonincreasing, then, $\varphi(v) = 1/\sqrt{B}$, for some $v = v_B$.

$$\varphi(\sqrt{B}u) < \frac{1}{\sqrt{B}} \Leftrightarrow \sqrt{B}u > v_B \Leftrightarrow u > \frac{v_B}{\sqrt{B}}.$$

Let us compute the value of v_B

$$\frac{\exp(-v_B^2)}{v_B} = \frac{1}{\sqrt{B}},$$

one has

$$1 + v_B^2 < \exp v_B^2 = \frac{\sqrt{B}}{v_B},$$

this implies that

$$v_B + v_B^3 < \sqrt{B} \Rightarrow v_B < B^{1/6}.$$

$$\varphi(\sqrt{B}u) < \varphi(v_B) \Leftrightarrow v_B < \sqrt{B}u.$$

This is true if

$$B^{1/6} < \sqrt{B}u,$$

so

$$u > \frac{B^{1/6}}{B^{1/2}} = B^{-1/3}.$$

Here

$$B = \frac{A+1}{2}; \qquad A = k^N.$$

So, it suffices that

$$u > \left(\frac{k^N+1}{2}\right)^{-1/3}.$$

On the other hand

$$u = \cos \pi a \Rightarrow a < \frac{1}{\pi} \arccos\left(\frac{k^N+1}{2}\right)^{-1/3}.$$

Then, one has

$$\int_0^a (\sin \pi u)^A \, du \le \frac{1}{\pi(A+1)}, \qquad \text{for } a < \frac{1}{\pi} \arccos\left(\frac{A+1}{2}\right)^{-1/3}.$$

So, one has

$$\int_0^{(1/\pi)\arccos((k^N+1)/2)^{-1/3}} (\sin \pi u)^{k^N} \, du \le \frac{1}{\pi(k^N+1)},$$

and

$$\int_{(1/\pi)\arccos((k^N+1)/2)^{-1/3}}^{1/2} (\sin \pi u)^{k^N} \, du \le \frac{1}{2} - \frac{1}{\pi} \arccos\left(\frac{k^N+1}{2}\right)^{-1/3}.$$

But

$$\sin\left(\frac{\pi}{2} - \arccos \alpha\right) = \alpha \simeq \frac{\pi}{2} - \arccos \alpha, \qquad \text{as } \alpha \to 0.$$

Hence for k large enough one has

$$\int_{(1/\pi)\arccos((k^N+1)/2)^{-1/3}}^{1/2} (\sin \pi u)^{k^N}\, du \le \frac{1}{\pi}\left(\frac{k^N+1}{2}\right)^{-1/3}.$$

In conclusion

$$\int_k^{k+1} \varphi(t)\, dt \le 2(k+1)\left[\frac{1}{\pi(k^N+1)} + \frac{1}{\pi}\left(\frac{k^N+1}{2}\right)^{-1/3}\right].$$

In the case where $1 - N/3 < -1$ (i.e. $N > 6$), the series with general term $\int_k^{k+1} \varphi(t)\, dt$ converges, consequently $(1/r)\int_0^r \varphi(t)\, dt \to 0$, as $r \to +\infty$. This ends the proof.

3. THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

3.1. The linear bounded case

We consider the following problem

$$\frac{dx}{dt}(t) = Ax(t) + f(t), \tag{3}$$

and the following hypotheses:

(H_1) Suppose that $A: E \to E$ is a dissipative, bounded linear operator;

(H_2) Suppose that $f: \mathbb{R} \to E$ is bounded, continuous and pseudo almost periodic.

THEOREM 3.1. Under the hypotheses (H_1) and (H_2), equation (3) has one and only one bounded solution which is pseudo almost periodic.

Proof. First we prove the uniqueness. Assume that equation (3) has two bounded solutions x_1 and x_2. Then $v = x_1 - x_2$ is a bounded solution for the linear equation

$$\frac{dx}{dt}(t) = Ax(t). \tag{4}$$

Let us note that the Dini derivative

$$D^- v(t) = \overline{\lim_{h \to 0^-}} \frac{v(t+h) - v(t)}{h}, \tag{5}$$

then one has

$$\tfrac{1}{2}\|D^- v(t)\|^2 = \langle Av(t), v(t)\rangle_- \le -c\|v(t)\|^2. \tag{6}$$

By integrating the differential inequality (6), we have

$$\|v(t)\| \le \exp(-c(t-a))\|v(a)\| \qquad \text{for } t > a. \tag{7}$$

Since v is bounded, we can take a near infinity and consequently we have $v(t) = 0 \Rightarrow v \equiv 0$, which gives the uniqueness property. ■

In the sequel we need the following result.

Remark 3.1. If A is dissipative then

$$\|\exp(tA)\| \leq \exp(-ct), \qquad \text{for } t \geq 0.$$

Indeed for $v_0 \in E$, let v be a solution of equation (4) with initial value v_0. By integrating the differential equation (4) we find that

$$\|(\exp tA)v_0\| \leq \exp(-ct)\|v_0\| \qquad \text{for all } t \geq 0.$$

So

$$\|\exp(tA)\| \leq \exp(-ct) \qquad \text{for all } t \geq 0.$$

We now continue the proof by establishing the existence. The only bounded solution of equation (3) is given by

$$x(t) = \int_{-\infty}^{t} \exp(A(t-s))f(s)\,ds.$$

Since

$$\|\exp tA\| \leq \exp(-ct), \qquad \text{for all } t \geq 0,$$

then, x is defined and furthermore

$$\|x\| \leq \|f\| \int_{-\infty}^{t} \exp(-c(t-s))\,ds,$$

$$\leq \|f\| \frac{1}{c} [\exp(-c(t-s))|_{-\infty}^{t}] \leq \|f\| \frac{1}{c}$$

x is a solution for equation (3). If $f(t) = g(t) + \varphi(t)$, where

$$g \in \mathcal{AP}(\mathbb{R}, E), \text{ and } \varphi \in \tilde{\mathcal{P}}\mathcal{AP}_0(\mathbb{R}, E),$$

then

$$x(t) = \int_{-\infty}^{t} \exp A(t-s)f(s)\,ds$$

$$= \int_{-\infty}^{t} \exp A(t-s)g(s)\,ds + \int_{-\infty}^{t} \exp A(t-s)\varphi(s)\,ds.$$

Let

$$u(t) = \int_{-\infty}^{t} \exp A(t-s)g(s)\,ds \qquad \text{and} \qquad v(t) = \int_{-\infty}^{t} \exp A(t-s)\varphi(s)\,ds.$$

Then u is an almost periodic function. Indeed, g is almost periodic, using Definition 2.1. Then, for $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $\rho \in \mathbb{R}$, there exists $\tau \in [\rho, \rho + \eta]$ with

$$\sup_{t \in R} \|g(t+\tau) - g(t)\| \leq \varepsilon.$$

It follows that

$$\sup_{t \in R} \|v(t+\tau) - v(t)\| \leq \frac{\varepsilon}{c}.$$

So v is almost periodic.

In order to show that x is in $\widetilde{\mathcal{PAP}}(\mathbb{R}, E)$, we need to show that $u \in \widetilde{\mathcal{PAP}}_0(\mathbb{R}, E)$, i.e. we need to show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|u(t)\| \,\mathrm{d}t = 0. \tag{8}$$

We have

$$\begin{aligned}
0 &\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|u(t)\| \,\mathrm{d}t \\
&\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \int_{-\infty}^{t} \exp(-c(t-s))\|\varphi(s)\| \,\mathrm{d}s\,\mathrm{d}t \\
&= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t\left(\int_{-\infty}^{-r} \exp(-c(t-s))\|\varphi(s)\| \,\mathrm{d}s\right) \\
&\quad + \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t\left(\int_{-r}^{t} \exp(-c(t-s))\|\varphi(s)\| \,\mathrm{d}s\right) \\
&= \underbrace{\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t\left(\int_{-r}^{t} \exp(-c(t-s))\|\varphi(s)\| \,\mathrm{d}s\right)}_{J_1} \\
&\quad + \underbrace{\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t \int_{-\infty}^{-r} \exp(-c(t-s))\|\varphi(s)\| \,\mathrm{d}s.}_{J_2}
\end{aligned}$$

Now

$$\begin{aligned}
J_1 &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \,\mathrm{d}t \int_{-r}^{t} \exp(-c(t-s)) \,\mathrm{d}s, \\
&= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \,\mathrm{d}t(c\exp(-c(t-s))|_{-r}^{t}, \\
&= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \,\mathrm{d}t\left(\frac{1}{c}[1-\exp(-c(t+r))]\right).
\end{aligned}$$

Since $-r \le t \le r$ and $c > 0$, then $(1/c)(1 - \exp(-c(t+r))$ is bounded. Furthermore, $\varphi \in \widetilde{\mathcal{PAP}}_0(\mathbb{R}, E)$, then $J_1 = 0$.

Also

$$J_2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{d}t \int_{-\infty}^{-r} \|\varphi(s)\| \exp(-c(t-s)) \,\mathrm{d}s.$$

By using the Fubini theorem, one has

$$J_2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-\infty}^{-r} \exp(cs)\|\varphi(s)\|\, ds \int_{-r}^{r} \exp(-c(t))\, dt,$$

$$= \lim_{r\to\infty} \frac{1}{2r} \|\varphi\| \int_{-\infty}^{-r} \exp(cs)\, ds \left(\frac{1}{c}[\exp(-cr) - \exp(cr)]\right),$$

$$= \lim_{r\to\infty} \frac{1}{2r} \|\varphi\| \left(\frac{1}{c^2}[1 - \exp(-2cr)]\right).$$

It is clear that $J_2 = 0$. This completes the proof of Theorem 3.1. ■

Remark 3.2. Looking to the quantity J_2, we can assume that φ is unbounded, in this case the end of proof is as follows

$$J_2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} dt \int_{-\infty}^{-r} \|\varphi(s)\| \exp(-c(t - s))\, ds,$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\|\, dt \int_{-r}^{r} \exp(-c(t - s))\, ds,$$

$$= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\|\, dt\left(\frac{1}{c}[\exp(-c(t - r)) - \exp(-c(t + r))]\right),$$

$$= \frac{1}{c} \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-c(t - r))\, dt$$

$$- \frac{1}{c} \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-c(t + r))\, dt.$$

If we consider the quantity

$$h(r) = \frac{1}{2r} \int_{r}^{\infty} \|\varphi(t)\| \exp(-c(t - r))\, dt,$$

and, integrate by parts, we obtain

$$h(r) = \frac{1}{2r}\left[\left(\int_{0}^{t} \|\varphi(u)\|\, du\right) \exp(-c(t - r))\right]_{r}^{+\infty}$$

$$+ \frac{c}{2r} \int_{r}^{\infty} \left(\int_{0}^{t} \|\varphi(u)\|\, du\right) \exp(-c(t - r))\, dt.$$

It is clear that the first term in the above equality tends to zero as $r \to +\infty$ and, by hypothesis $\varphi \in \tilde{\mathcal{P}}\mathcal{AP}_0(\mathbb{R})^n$, we have

$$\frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\|\, dt \to 0 \qquad \text{as } r \to \infty.$$

So, we have that there exists $\varepsilon(r)$ such that

$$\int_0^t \|\varphi(u)\| \, du \le \varepsilon(r)t.$$

Now, one has

$$\begin{aligned}
&\frac{c}{2r}\int_r^\infty \left(\int_0^t \|\varphi(u)\| \, du\right) \exp(-c(t-r)) \, dt \\
&\le \frac{\varepsilon(r)c}{r}\int_r^\infty t\exp(-c(t-r)) \, dt \\
&= \left[\frac{\varepsilon(r)c}{r}\frac{-1}{c}t\exp(-c(t-r))\right]_r^{+\infty} - \frac{\varepsilon(r)c}{r}\int_r^\infty \exp(-c(t-r)) \, dt \\
&= \frac{\varepsilon(r)c}{r}\frac{+1}{c}r - \frac{1}{c}\frac{\varepsilon(r)c}{r} = \varepsilon(r)\left(1-\frac{1}{r}\right).
\end{aligned}$$

This last quantity goes to zero as r goes to infinity. By the same arguments, we prove that the quantity

$$-\frac{1}{c}\frac{1}{2r}\int_r^\infty \|\varphi(t)\| \exp(-c(t+r)) \, dt$$

goes to zero as r goes to infinity.

Consequently, we have that $J_2 \to 0$ as $r \to \infty$.

3.2. The unbounded linear case

In the sequel we consider the following equation

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(t), \\ x_0 \in D(A), \end{cases} \tag{9}$$

where A is an unbounded linear operator, and the following hypotheses:

(H_3) Suppose that A is an infinitesimal generator of a stable C_0-semi group, $(S(t))_{t\ge 0}$ on the Banach space X, that is, there exist constants M, $c > 0$ such that

$$\|S(t)\| \le M\exp(-ct), \qquad \text{for } t \ge 0.$$

(H_4) Suppose that $f\colon \mathbb{R} \to X$ is p.a.p., f is continuously differentiable and

$$\sup_{t\in\mathbb{R}} \left\|\frac{df}{dt}(t)\right\| < \infty.$$

THEOREM 3.2. Suppose that (H_3) and (H_4) hold. Then equation (9) has one and only one bounded solution which is pseudo almost periodic.

For the sequel we need the following propositions.

PROPOSITION 3.3 (Zaidman [8]). Let X be a Banach space and let $(S(t))_{t\geq 0}$ be a one-parameter semigroup of class C_0, satisfying the following inequality

$$\|S(t)\| \leq M\,\mathrm{e}^{-ct}, \qquad t \geq 0$$

where $M > 0$ and $c > 0$. Let A be the infinitesimal generator of $S(t)$ and let $u\colon \mathbb{R} \to D(A)$ be a solution of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax. \tag{10}$$

Then, if one has $\sup_{t\in\mathbb{R}} \int_t^{t+1} |u(s)|^2\, \mathrm{d}s < \infty$, it follows that $u \equiv 0$.

PROPOSITION 3.4 (Zaidman [8]). Assume that f is C^1. In that case, for any element $x_0 \in D(A)$, the solution u of equation (9) is given by

$$u(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s.$$

For the proof of these propositions we refer to [8].

Remark 3.3. If u is a bounded solution of (10), then the condition of proposition 4.1 is satisfied.

Proof of the theorem. Uniqueness. Let u_1 and u_2 be two bounded solutions of equation (9), then $v = u_1 - u_2$ is a bounded solution of equation (10). By Proposition 4.1, $v = 0$. So, $u_1 = u_2$.

Existence. Let u be the function defined by

$$u(t) = \int_{-\infty}^t S(t-s)f(s)\,\mathrm{d}s = \lim_{R\to+\infty} \int_{-R}^t S(t-s)f(s)\,\mathrm{d}s.$$

We remark that $\sigma \to S(t-\sigma)f(\sigma)$ is continuous over $[R, t]$, then, it is Riemann integrable, which implies that $\int_{-R}^t S(t-s)f(s)\,\mathrm{d}s$ is defined. Let us prove that u is a solution of equation (9). We have

$$u(t) = \lim_{n\to\infty} \int_{-n}^t S(t-s)f(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_0^{t+n} S(s)f(t-s)\,\mathrm{d}s.$$

Let $x = t + n$. We define the function $v_n(x)$ by

$$v_n(x) = \int_0^x S(s)g_n(x-s)\,\mathrm{d}s,$$

where $g_n(x) = f(x-n)$. It follows that g_n is C^1. By Proposition 3.4,

$$v_n(x) = \int_0^x S(s)g_n(x-s)\,\mathrm{d}s \qquad \text{and} \qquad v_n(0) = 0$$

and v_n satisfies

$$\frac{dv_n}{dt} = Av_n + g_n.$$

Taking $x = t + n$, one has

$$\frac{dv_n}{dt} = Av_n + f(t).$$

Then, v_n is a solution of equation (9), and

$$v_n(t) = \int_0^{t+n} S(s)f(t - s)\,ds.$$

So,

$$\frac{dv_n}{dt} = \int_0^{t+n} S(s)\frac{d}{dt}f(t - \sigma)\,d\sigma + S(t + n)f(-n).$$

We have also that

$$\|S(t + n)f(-n)\| \leq M\exp(-ct)\exp(-cn)\|f\|_\infty \underset{n\to+\infty}{\to} 0.$$

Henceforth, $\lim_{n\to\infty}(dv_n/dt)$ exists and is equal to $\int_0^\infty S(s)(d/dt)f(t - \sigma)\,d\sigma$ uniformly on any bounded subset of $\mathbb{R}$.

$$\frac{dv_n}{dt} = Av_n + f(t) \Rightarrow Av_n = \frac{dv_n}{dt} - f(t)$$

$$v_n(t) \to u(t), \qquad Av_n = u(t) - f(t).$$

Since A is a closed operator, then

$$u(t) \in D(A) \qquad \text{and} \qquad Au(t) = v(t) - f(t).$$

One has

$$\frac{d}{dt}v_n(t) \underset{n\to\infty}{\to} v(t), \quad \text{uniformly on any bounded set of } \mathbb{R},$$

$$v_n \to u(t), \quad \text{uniformly on any bounded set of } \mathbb{R}.$$

Hence, u is differentiable and $(d/dt)u(t) = v(t)$. So

$$Au = \frac{d}{dt}u(t) - f(t), \qquad \text{for } t \in \mathbb{R}.$$

Consequently, u is a solution of equation (9) which is bounded. To end the proof, it suffices to show that u is pseudo almost periodic. The bounded solution of equation (9) is given by

$$u(t) = \int_{-\infty}^{t} S(t - s)f(s)\,ds.$$

Since

$$\|S(t)\| \leq M\exp(-ct), \qquad \text{for all } t \geq 0,$$

then, x is defined and furthermore, we have

$$\|x\| \leq \|f\| \int_{-\infty}^{t} M\exp(-c(t - s))\,\mathrm{d}s,$$

$$\leq \|f\| \frac{M}{c} [\exp(-c(t - s))|_{-\infty}^{t}] \leq \|f\| \frac{M}{c},$$

u is a solution of equation (9).

If $f(t) = g(t) + \varphi(t)$, where $g \in \mathcal{AP}(\mathbb{R}, E)$, and $\varphi \in \tilde{\mathcal{P}}\mathcal{AP}_0(\mathbb{R}, E)$. Then

$$u(t) = \int_{-\infty}^{t} S(t - s)f(s)\,\mathrm{d}s = \int_{-\infty}^{t} S(t - s)g(s)\,\mathrm{d}s + \int_{-\infty}^{t} S(t - s)\varphi(s)\,\mathrm{d}s.$$

Let

$$v(t) = \int_{-\infty}^{t} S(t - s)g(s)\,\mathrm{d}s \qquad \text{and} \qquad w(t) = \int_{-\infty}^{t} S(t - s)\varphi(s)\,\mathrm{d}s.$$

Then $v(t)$ is an almost periodic function. Indeed, g is almost periodic, using Definition 2.1. Then, for $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $\rho \in \mathbb{R}$, there exists $\tau \in [\rho, \rho + \eta]$ and

$$\sup_{t \in R} \|g(t + \tau) - g(t)\| \leq \varepsilon,$$

which gives that

$$\sup_{t \in R} \|v(t + \tau) - v(t)\| \leq \frac{M\varepsilon}{c}.$$

So v is almost periodic.

In order to show that u is in $\tilde{\mathcal{P}}\mathcal{AP}(\mathbb{R}, E)$, we need to show that $w(t) \in \tilde{\mathcal{P}}\mathcal{AP}_0(\mathbb{R}, E)$, i.e. we need to show that

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|w(t)\|\,\mathrm{d}t = 0. \tag{11}$$

The proof is similar to the one given in Theorem 3.1. This ends the proof of Theorem 3.2.

4. EXAMPLES

Consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(t) \tag{12}$$

where $f\colon \mathbb{R} \to H$, where H is a Hilbert space, $t \to f(t)$ is continuous, bounded, pseudo almost periodic, and $A\colon H \to H$, defined by $Ax = -x$.

We have $(Ax, x) = -\|x\|^2$, so A is dissipative; using Theorem 3.1 we have that equation (12) has only one bounded solution which is pseudo almost periodic.

We consider the following equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -\Delta u + f(t, x), & t \geq 0,\ x \in \mathbb{R}^n \\ u(0, x) = u_1(x), & \dfrac{\partial u}{\partial t}(0, x) = u_2(x), \end{cases} \tag{13}$$

where

$$u_1 \in H^1(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) \,\middle|\, \frac{\partial v}{\partial x_i} \in L^2(\mathbb{R}^n) \right\}$$

where $\partial v/\partial x_i$ is a distribution derivative $u_2 \in L^2(\mathbb{R}^n)$.

$f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying that $f = g + \varphi$, with g is almost periodic in t uniformly with respect to x, $g(t, \cdot) \in L^2(\mathbb{R}^n)$, $\varphi(t, \cdot) \in L^2(\mathbb{R}^n)$

$$G\colon \mathbb{R} \to L^2(\mathbb{R}^n)$$

$$t \to g(t, \cdot)$$

is almost periodic

$$\Psi\colon \mathbb{R} \to L^2(\mathbb{R}^n)$$

$$t \to \varphi(t, \cdot)$$

satisfies the ergodicity property that is

$$\frac{1}{2r} \int_{-r}^{+r} \|\Psi(s)\|_{L^2(\mathbb{R}^n)}\, ds \to 0, \qquad \text{as } r \to +\infty.$$

So the problem (13) is equivalent to the following

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} \tag{14}$$

with the variables change $u_2 = \partial u_1/\partial t$.

Henceforth, the phase space is $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

PROPOSITION 4.1 (Pazy [9]). The operator

$$A = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}$$

is the infinitesimal generator of a C_0 semigroup $S(t)$ on the space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying the following inequality

$$\|S(t)\| \leq 4 \exp(-2t), \qquad t \geq 0$$

$$D(A) = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n).$$

So if we put $V = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then, the problem (14) becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} V = AV + F(t), & t \geq 0 \\ V(0) = V_0. \end{cases} \tag{15}$$

PROPOSITION 4.2. If F is a continuously differentiable function such that

$$\sup_{t \in \mathbb{R}} \left\| \frac{\mathrm{d}F(t)}{\mathrm{d}t} \right\| < \infty,$$

then equation (15) has one and only one bounded solution W which is pseudo almost periodic solution.

Remark 4.1. In this case the function that is defined by $v(t, x) = W(t)(x)$ satisfies the problem (13) and is pseudo almost periodic in $L^2(\mathbb{R}^n)$ norm.

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