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PERIODIC AND ALMOST PERIODIC RESULTS FOR SOME DIFFERENTIAL EQUATIONS IN BANACH SPACES

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INTRODUCTION

The problem of existence of periodic and almost periodic solutions of functional differential equations has been considered by many authors. There are many papers in this area. We can quote the works of [1-8].

In finite dimensions, for linear periodically forced equations of the type

$$\frac{dx}{dt}(t) = A(t)x(t) + f(t) \tag{1}$$

the following alternative is well known.

If the homogeneous equation

$$\frac{dx}{dt}(t) = A(t)x(t)$$

has $x = 0$ as its only periodic solution then equation (1) has one and only one periodic solution for each periodic forcing f.

Extensions of this result have been obtained in three directions:

(1) from finite top infinite dimension;
(2) from periodic to almost periodic forcing;
(3) from linear to nonlinear equations.

For periodic forcing in infinite dimensions in the nonlinear situation one of the classic results was described by Perov and Trubnikov in a series of papers. Apart from a Hilbert structure, the main assumption made by Perov and Trubnikov is a strong dissipativeness condition. This assumption, quite natural in many applications has also been considered in nonHilbert settings.

Amongst the authors who worked in this direction, we can quote Ait Dads [1], Hanebaly [3], who deals with a more general dissipativeness condition. These authors have extended to nonlinear almost periodic equations the above mentioned alternative. The method developed in [1, 3] consists of two steps.

First, to prove existence and uniqueness of a bounded solution of the equation; then, using a suitable characterization of almost periodicity due to Bochner, to show that the bounded solution is almost periodic.

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Recently, Kato and Imai did some work in that direction. In [9, 10], they deal with the following differential equation

$$\frac{dx}{dt}(t) = A(t, x(t)) + f(t) \tag{2}$$

aiming at extending an old existence result obtained in [6] by using dissipative type conditions on A.

Let us recall the essential steps of these two papers.

Throughout $\mathbb{R}^n$ is, as usual, the space of n-tuples of reals, $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$.

$$A: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

$$f: \mathbb{R} \to \mathbb{R}^n.$$

We denote

$$[\ ,\ ]: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

the function defined by

$$[x, y] = \lim_{h \to 0^+} \frac{1}{h}(\|x + hy\| - \|x\|).$$

Assuming the following conditions;

(K_1) $A(t, x)$ is a continuous mapping.

(K_2) $f(t)$ is a continuous mapping and $\|f(t) + A(t, 0)\| \leq N$ for all $t \in \mathbb{R}$, where N is a positive constant.

(K_3) There exists $p \in C(\mathbb{R}, \mathbb{R})$, such that for some positive constants δ, γ, T_0 the following two properties hold:

(i) $$p(t) \leq -\delta \qquad (t \in]-\infty, T_0])$$

and

(ii) $$\lim_{t \to +\infty} \frac{1}{t - s} \int_s^t p(\sigma)\, d\sigma = -\gamma \qquad \text{(uniformly for } s > T_0\text{)}.$$

(K_4) For all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$

$$[x - y, A(t, x) - A(t, y)] \leq p(t)\|x - y\|.$$

(K_4) Is a strong dissipativity condition.

The main result is given in the following theorem.

THEOREM 0.1 [9]. Suppose that conditions (K_1)–(K_4) are satisfied. Suppose, furthermore, that $A(t, x)$ is almost periodic in t uniformly for $x \in \mathbb{R}^n$ and f is almost periodic. Then, equation (2) has a unique almost periodic solution on $\mathbb{R}$.

In this paper, we relate the results obtained by Kato and Imai to those obtained previously by [1, 3]. We show that the methods introduced in [1, 3] apply to the situation envisaged here. In fact, they allow to handle a more general setting than in [9, 10], since our results are stated in an arbitrary Banach space. As a preliminary, a discussion of the assumptions introduced in [9, 10], is made.

In particular, the main assumption, as regards the novelty in Kato and Imai's papers, that is (K_3), is reformulated and a more general assumption, along the lines of (K_3), is shown to suffice.

The proof of this result is presented as follows.

One proves the uniqueness of the Cauchy problem, which gives the uniqueness of bounded solutions defined over $\mathbb{R}$. After this, one shows that the bounded solutions are asymptotically equal at infinity, that is, the distance between two solutions is going to zero (this affirmation is given in Lemma 1.2). Using this last result the proof of the periodicity or almost periodicity of the bounded solution is given.

1. PRELIMINARIES

Let $(E, \|\cdot\|)$ be a Banach space, and let us consider equation (2) with

$$A: \mathbb{R} \times E \to E$$

$$f: \mathbb{R} \to E.$$

(1) Hypothesis (K_3) implies the following.

(H_3) Suppose that there exist $p \in C(\mathbb{R}, \mathbb{R})$ and positive constants δ, δ_1, T_0 and T_1 such that

$$p(t) \leq -\delta \qquad t \in]-\infty, -T_0]$$

$$p(t) \leq -\delta_1 \qquad t \in [T_1, +\infty[.$$

In fact part (ii) of hypothesis (K_3) implies there exists T_1, $T_1 > T_0$ such that $\forall t > T_1$ and $s \geq T_1$, one has

$$\frac{1}{t-s} \int_s^t p(\sigma)\, \mathrm{d}\sigma \leq -\frac{\gamma}{2},$$

then, letting s tends to t, we obtain

$$p(t) \leq -\frac{\gamma}{2} = -\delta_1.$$

The next two lemmas correspond to Lemmas 2.2 and 2.3 in [9], assuming (H_3) instead of (K_3).

Lemma 1.1. Suppose that (H_3) is satisfied. Then,

$$\int_{-T_0}^{t} p(\sigma)\, \mathrm{d}\sigma \to -\infty \qquad \text{as } t \to +\infty$$

and,

$$\sup\left\{\int_{-T_0}^{t} \exp\left(\int_s^t p(\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s;\ t \geq -T_0\right\} < \infty.$$

Proof. It follows from (H_3)

$$\int_{-T_0}^{t} p(\sigma)\,d\sigma = \int_{-T_0}^{T_1} p(\sigma)\,d\sigma + \int_{T_1}^{t} p(\sigma)\,d\sigma$$

$$\leq \int_{-T_0}^{T_1} p(\sigma)\,d\sigma - \delta_1(t - T_1) \to -\infty \qquad \text{as } t \to +\infty.$$

Since

$$\int_{s}^{t} p(\sigma)\,d\sigma \leq -\delta_1(t - s) \qquad \text{for } t \geq s \geq T_1,$$

we have, for each $t \geq T_1$

$$\int_{-T_0}^{t} \exp \int_{s}^{t} (p(\sigma)\,d\sigma)\,ds = \int_{-T_0}^{T_1} \exp \int_{s}^{t} (p(\sigma)\,d\sigma)\,ds + \int_{T_1}^{t} \exp \int_{s}^{t} (p(\sigma)\,d\sigma)\,ds$$

$$= \int_{-T_0}^{T_1} \exp \int_{s}^{T_1} (p(\sigma)\,d\sigma)\,ds \exp \int_{T_1}^{t} (p(\sigma)\,d\sigma)\,ds$$

$$+ \int_{T_1}^{t} \exp \int_{s}^{t} (p(\sigma)\,d\sigma)$$

$$\leq \int_{-T_0}^{T_1} \exp \int_{s}^{T_1} (p(\sigma)\,d\sigma)\,ds + \int_{T_1}^{t} \exp[-\sigma_1(t - s)]\,ds$$

$$= \int_{-T_0}^{T_1} \exp\left(\int_{s}^{T_1} p(\sigma)\,d\sigma\right) ds + \frac{1}{\delta_1}\{1 - \exp[-\delta_1(t - T_1)]\}.$$

Therefore,

$$\int_{-T_0}^{t} \exp \int_{s}^{t} (p(\sigma)\,d\sigma)\,ds \leq \int_{-T_0}^{T_1} \exp \int_{s}^{T_1} (p(\sigma)\,d\sigma)\,ds + \frac{1}{\delta_1}$$

for all $t \geq -T_0$.

LEMMA 1.2. Suppose that (K_4) is satisfied. Let $u(t)$ and $v(t)$ be solutions of equation (2) on an interval $[a, b]$. Then

$$\|u(t) - v(t)\| \leq \|u(a) - v(a)\| \exp\left(\int_{a}^{t} p(\sigma)\,d\sigma\right) \tag{3}$$

for all $t \in [a, b]$.

The proof is similar to the one given in Lemma 2.3 in [9]. It is not related to the dimension of the space.

2. EXISTENCE OF BOUNDED SOLUTIONS

Suppose that (K_1) and (H_3) are satisfied. Let N, δ and T_0 be positive constants as given in (K_1) and (H_3), respectively. Set $r_0 = N/\delta$,

$$\Gamma = \max\left\{1, \sup_{-T_0 \le s \le t} \exp \int_s^t p(\sigma)\,\mathrm{d}\sigma, \sup_{-T_0 \le t} \int_{-T_0}^t \exp \int_s^t (p(\sigma)\,\mathrm{d}\sigma)\,\mathrm{d}s\right\} \tag{4}$$

and

$$r = \Gamma(r_0 + N). \tag{5}$$

We remark that r is finite by Lemma 1.1. Then, we have the following theorem.

THEOREM 2.1. Suppose that conditions (K_1), (K_2), (H_3) and (K_4) are satisfied. Let r be as defined by (5). Then, equation (2) has a unique bounded solution $u(t)$ on $\mathbb{R}$, $\|u(t)\|_E \le r$ for $t \in \mathbb{R}$. Furthermore, if v is any solution of equation (2), then $\|u(t) - v(t)\| \to 0$ as $t \to \infty$.

Proof. The first part of this is similar to the one given in [1] by considering the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t, x(t)) + f(t), \qquad x(-n) = u_0 \in E. \tag{6}$$

Then (6) has a unique solution u_n on $[-n, +n]$.

Using Lemma 1.2, we can prove directly that the sequence (u_n) is a uniform Cauchy sequence in every bounded subset of $\mathbb{R}$. Indeed, let n and m be two positive integers such that $m \ge n$. Then, u_n and u_m are defined on $[-n, n]$ and, by inequality (3) in Lemma 1.2, for $t \in [-n, n]$, we have

$$\|u_m(t) - u_n(t)\| \le \|u_m(-n) - u_n(-n)\| \exp\left(\int_{-n}^t p(\sigma)\,\mathrm{d}\sigma\right).$$

Let

$$\alpha = \|u_m(-n) - u_n(-n)\|.$$

So, if $-n \le t \le -T_0$, then

$$\exp\left(\int_{-n}^t p(\sigma)\,\mathrm{d}\sigma\right) \le \exp(-\delta)(t + n),$$

which yields that (u_n) is a uniform Cauchy sequence in every bounded subset of $\mathbb{R}$.

If $-n \le -T_0 \le t$, then

$$\exp\left(\int_{-n}^t p(\sigma)\,\mathrm{d}\sigma\right) = \exp\left(\int_{-n}^{-T_0} p(\sigma)\,\mathrm{d}\sigma\right) \exp\left(\int_{-T_0}^t p(\sigma)\,\mathrm{d}\sigma\right)$$

$$\le \exp\left(\int_{-T_0}^t p(\sigma)\,\mathrm{d}\sigma\right) \exp(-\delta)(-T_0 + n).$$

So,

$$\exp\left(\int_{-T_0}^{t} p(\sigma)\,\mathrm{d}\sigma\right)\exp(-\delta)(-T_0+n)\to 0 \qquad \text{as } n\to\infty$$

which implies that (u_n) is a uniform Cauchy sequence in every bounded subset of $\mathbb{R}$. So, its limit is a bounded solution of equation (2) over $\mathbb{R}$, which ends the proof. ∎

3. ALMOST PERIODICITY PROPERTY

In this section, we begin by the *Bochner characterization of almost periodicity*.

THEOREM 3.1 [1]. Let $f \in C(\mathbb{R}\times E, E)$. Then $f(t,x)$ is an almost periodic function in t uniformly with respect to x in E, if for any real sequences θ_n, σ_n, there exist two subsequences θ'_n, σ'_n such that:

$$f(t+\theta'_n, x) \text{ converges to a function } g(t,x)$$

and

$$f(t+\theta'_n+\sigma'_n, x) \text{ and } g(t+\sigma'_n, x) \text{ converge to a same limit } h(t,x).$$

Furthermore, $g(t,x)$ is also almost periodic in t uniformly with respect to x in E.

Now, we are in a position to give the main result of this section.

THEOREM 3.2. Suppose that the hypotheses (H_3) and (K_4) are satisfied. Suppose, furthermore, that $A(t,x)$ is almost periodic in t uniformly for $x\in E$ and $f(t)$ is almost periodic in t. Then, equation (2) has a unique almost periodic solution on $\mathbb{R}$.

Proof. Without any loss of generality we can assume that $f\equiv 0$. Note that, in view of Theorem 3.1, for each sequence $\sigma_n\in\mathbb{R}$, there exists a subsequence noted by θ_n, such that the sequence

$$A_n(t,x) = A(t+\theta_n, x)$$

converges in $\mathbb{R}\times E$, with as a limit a function $B(t,x)$, the convergence being uniform in $\mathbb{R}\times K$, where K is any compact subset of E. $B(t,x)$ is almost periodic in t uniformly with respect to x in compact subsets of E. Clearly assumptions (K_1) and (K_2) are satisfied by B. (H_3) and (K_4) are supposed to be true. So, Theorem 2.1 applies and yields that equation (2) has a unique bounded solution defined over $\mathbb{R}$. Let $u(t)$ denote this solution. It is easy to verify that the function B satisfies the same conditions as A. So, the limit equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = B(t,y), \tag{7}$$

has a unique bounded solution defined over $\mathbb{R}$. Let u_0 be this solution and consider the sequence (u_n) defined by $u_n(t)=u(t+\theta_n)$; u_n is a solution of the following equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A_n(t,z) = A(t+\theta_n, z).$$

We will show that for all $t \in \mathbb{R}$, $u_n(t)$ converges to $u_0(t)$ in E. In fact, put

$$v_n(t) = \|u_n(t) - u_0(t)\|.$$

$$D_+\|f(t)\| = [f(t), f'(t)].$$

Then,

$$\begin{aligned}
D_+ v_n(t) &= \left[u_n(t) - u_0(t), \frac{\mathrm{d}}{\mathrm{d}t}(u_n(t) - u_0(t)) \right] \\
&= [u_n(t) - u_0(t), A(t + \theta_n, u_n(t)) - B(t, u_0(t))] \\
&= [u_n(t) - u_0(t), A(t + \theta_n, u_n(t)) - A(t + \theta_n, u_0(t)) \\
&\quad + A(t + \theta_n, u_0(t)) - B(t, u_0(t))] \\
&\le [u_n(t) - u_0(t), A(t + \theta_n, u_n(t)) - A(t + \theta_n, u_0(t))] \\
&\quad + [u_n(t) - u_0(t), A(t + \theta_n, u_0(t)) - B(t, u_0(t))] \\
&\le p(t)\|u_n(t) - u_0(t)\| + [u_n(t) - u_0(t), A(t + \theta_n, u_0(t)) - B(t, u_0(t))] \\
&\le p(t)(v_n(t)) + [u_n(t) - u_0(t), A(t + \theta_n, u_0(t)) - B(t, u_0(t))].
\end{aligned}$$

From this, we conclude that

$$D_+ v_n(t) \le p(t)v_n(t) + b_n(t), \tag{8}$$

where

$$b_n(t) = \|A(t + \theta_n, u_0(t)) - B(t, u_0(t))\|.$$

To prove this inequality, we have used the following property of the function [,]:

$$[x, y + z] \le [x, y] + \|z\|.$$

Then, integrating the differential inequality (8), we have the following inequality

$$v_n(t) \le \underbrace{\exp \int_{t_0}^{t} p(\sigma)\,\mathrm{d}\sigma \cdot v_n(t_0)}_{I} + \underbrace{\int_{t_0}^{t} \left(\exp \int_{s}^{t} p(\sigma)\,\mathrm{d}\sigma\right) b_n(s)\,\mathrm{d}s.}_{J}$$

Letting t_0 fixed and n go to $+\infty$, we obtain, thanks to the fact that when $n \to -\infty$, $J \to 0$, because $A(s + \theta_n, u_0(s))$ converges to $B(s, u_0(s))$ uniformly with respect to $s \in [t_0, t]$

$$\lim_{n \to +\infty} \sup v_n(t) \le M \exp\left(\int_{t_0}^{t} p(\sigma)\,\mathrm{d}\sigma\right) \qquad M = \sup_{t \in R} \|u(t) + u_0(+))\|.$$

This being for every $t_0 < t$, we arrive at

$$\lim_{n \to +\infty} \sup v_n(t) = 0.$$

Thus, we have proof that $u_n(t)$ converges to $u_0(t)$ for each $t \in \mathbb{R}$.

In order to conclude the proof of Theorem 3.2, we apply the second characterization of Bochner's theorem. Let θ_n, σ_n be two sequences in $\mathbb{R}$, we have to show that we can determine two subsequences θ_n', σ_n' such that:

$$u(t + \theta_n') \to v(t)$$

and

$$u(t + \theta_n' + \sigma_n') \text{ and } v(t + \sigma_n') \text{ converge to a same limit } w(t).$$

The subsequences can be determined as a consequence of the almost periodicity of $A(t, x)$. So, we have

$$A(t + \theta_n', x) \to B(t, x)$$

and

$$A(t + \theta_n' + \sigma_n', x) \text{ and } B(t + \sigma_n', x)$$

converge to a same function $C(t, x)$. Finally, applying the first part of the proof, we can see that $u(t + \theta_n')$ converges to the unique bounded solution v of the equation defined by B and $u(t + \theta_n' + \sigma_n')$ and $v(t + \sigma_n')$ converge to the unique solution $w(t)$ of the equation associated with C.

Remark. Hypothesis (K_3) as well as (H_3) yields existence of bounded solutions. It is not necessary for this to assume that the equation is almost periodic.

On the other hand, in the case the equation is almost periodic, then (K_3) as well as (H_3) read

$$[x - y, A(t,x) - A(t,y)] \leq -\delta\|x - y\|, \quad x, y \in E, \qquad \text{for every } t \in \mathbb{R}.$$

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