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A Note on "The Discrete Lyapunov Function..."

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INTRODUCTION

In [1, 2], the following question was considered: to prove the existence of sustained oscillations near a steady state for delay differential equations of monotone type. Our model equation was

$$x'(t) = -x(t) + x^2(t-\omega), \tag{1}$$

which is connected to a problem of electronic avalanche. Equation (1) is of monotone type when restricted to non-negative solutions. This means that for any pair of initial data φ, ψ, such that $0 \leqslant \varphi \leqslant \psi$, the solutions $x_t(\varphi)$ and $x_t(\psi)$ are in the same order: $x_t(\varphi) \leqslant x_t(\psi)$. For recent advances on monotone semiflows generated by functional differential equations, please refer to [7]. For our equation, oscillations of interest are those near the equilibrium point $x=1$. By a translation of the origin, (1) can be treated as a special case of the following class of delay differential equations,

$$x'(t) = f(x(t), x(t-1)), \tag{2}$$

where f is C^1, $f(0, 0) = 0$, and f is increasing with respect to the second variable. When f is of this type, we say that the equation has a positive feedback. These will be the basic hypotheses throughout the rest of the paper. Sustained oscillations refer to solutions oscillating near 0 whose oscillations do not damp out at infinity. Periodic solutions are of this type [3], but we are concerned here with more general types of oscillations.

It was proved in [1, 2] that existence of sustained oscillatory solutions is implied by the fact that 0 is not a homoclinic point.

The question of whether 0 may be a homoclinic point of an equation of type (2) was recently treated by Yulin Cao [5]. In fact, the main issue addressed in [5] is not homoclinic points. Two important properties are proved in [5]: (1) A count function for the zeros of a solution is shown to be non-decreasing along the solutions. This is the so-called discrete

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Lyapunov function [5, Theorem 1.5]. (2) A strong relation is established between the exponential decay of solutions and the number of zeros [5, Theorem 2.2]. (See Section 2 for more precise statements.) In the case of positive feedback, it is then concluded that 0 is not a homoclinic point.

In this short note, we would like first of all to exploit the above result in the problem of sustained oscillations: we prove that infinitely many such solutions can be exhibited. Our secondary purpose is to make some comments regarding the proof of [5, Theorem 2.2]. We propose a simplification which uses the same technique as the one presented in Cao's paper but concentrates on the main mechanism of the proof.

1. Sustained Oscillatory Solutions

We consider an equation of type (2) with the same general assumptions as those stated above. We denote

$$a = \frac{\partial f}{\partial x}(0, 0), \qquad b = \frac{\partial f}{\partial y}(0, 0).$$

Throughout most of our paper, $b > 0$, but a few results hold if b is just $\neq 0$. It is possible that zero might not be the only steady state, and indeed it is the case for our model equation (1). For that equation, let us recall that we consider oscillations near $x = 1$.

The state space for Eq. (2) is $\mathscr{C} = \mathscr{C}([-1, 0], \mathbb{R})$, the space of continuous functions from $[-1, 0]$ into $\mathbb{R}$. For $\varphi \in \mathscr{C}$, $\#(\varphi)$ denotes the number of zeros of φ ($= \infty$ if there are infinitely many zeros).

We need a result on the roots of the characteristic equation associated with Eq. (2). The linearization of E. (2) near zero is

$$x'(t) = ax(t) + bx(t-1), \tag{3}$$

whose characteristic equation is

$$\lambda = a + b \exp(-\lambda). \tag{4}$$

We introduce the numbers $I^-(a, b) = \inf\{\beta, \beta \geqslant 0$, such that $\alpha + i\beta$ is a root of (4) for some $\alpha < 0\}$. $I^+(a, b) = \sup\{\beta, \beta \geqslant 0$, such that $\alpha + i\beta$ is a root of (4) for some $\alpha > 0\}$. As a convention, we set $I^- = +\infty$, $I^+ = -\infty$, if the corresponding set is empty.

The following result was proved for the first time by Wright [8], in a more general setting. See also [5, Lemma 3.1].

Proposition 1. *For each $b > 0$, $a \in \mathbb{R}$, we have*

$$I^-(a, b) > I^+(a, b).$$

We now look at the discrete Lyapunov function, having in view relating it more explicitly to the count function $\#$.

LEMMA 2. *Assume that f is C^1, $f(0,0)=0$, and $f(0,y)\neq 0$ for $y\neq 0$. Let φ in $\mathscr{C}$, such that $\#(\varphi)<+\infty$. Let x be the solution of (2), such that $x_0=\varphi$. Then, $\#(x_t)<+\infty$, for each $t\geqslant 0$.*

Proof. Denote by τ the infimum of $t\geqslant 0$ such that $\#(x_t)=+\infty$. We want to show that $\tau=+\infty$. Assuming on the contrary that $\tau<+\infty$, we show first that near τ, x has only finitely many zeros. Denote $g(t)=f(0, x(t+\tau-1))$, $a(t)x=f(x, x(t+\tau-1))-f(0, x(t+\tau-1))$.

Equation (2) can be rewritten in the form

$$x'(t+\tau)=a(t)\,x(t+\tau)+g(t), \tag{5}$$

which, integrated from 0 to t, yields

$$x(t+\tau)=\int_0^t \exp\left(\int_s^t a(u)\,du\right) g(s)\,ds.$$

By the property defining τ, we may assert that $x(t)$ has at most a finite number of zeros near $\tau-1$ [on an interval $]\tau-1-\varepsilon,\ \tau-1+\varepsilon[$ (with $\varepsilon>0$) if $\tau>0$, or just a semi-interval $[\tau-1,\ \tau-1+\varepsilon[$ if $\tau=0$]. So, we can find a neighborhood $\mathscr{I}$ of 0 in $\mathbb{R}$ (or $\mathscr{I}\subseteq\mathbb{R}^+$ if $\tau=0$), such that the only zero of $x(t+\tau-1)$ in $\mathscr{I}$ is $t=0$. Therefore, in view of the assumption on $f(0,y)\neq 0$ for $y\neq 0$, we may conclude that the function g keeps a constant sign in each of the sets $\mathscr{I}\cap\mathbb{R}^-$ and $\mathscr{I}\cap\mathbb{R}^+$, so the only zero of $x(t+\tau)$ in $\mathscr{I}$ is $t=0$.

We can now conclude the proof of Lemma 2: suppose first that $\tau=0$. Then, from the above argument, we deduce that for $t>0$ small, x has no zero on $]0,t]$, which implies that $\#(x_t)\leqslant\#(\varphi)<+\infty$, in contradiction with the definition of τ. Suppose now that $\tau>0$. Then, we deduce from the above that there exists $\varepsilon>0$, such that x has no zero other than τ on $[\tau-\varepsilon,\ \tau+\varepsilon]$. Therefore, for $\tau\leqslant t\leqslant\tau+\varepsilon$, we have $\#(x_t)\leqslant\#(x_{\tau-\varepsilon})+1$ $<+\infty$, also in contradiction with the definition of τ. ∎

LEMMA 3. *Assume the same hypotheses as those in Lemma 2 for f. Let $\varphi\in\mathscr{C}$, such that $\#(\varphi)$ is finite. Let x be the solution of (2) such that $x_0=\varphi$. Then, the zeros of X on $[0,+\infty[$ are either a finite family or an increasing unbounded sequence (t_n).*

We may now restate [5, Theorem 1.5] as follows.

PROPOSITION 4. *Suppose the same hypotheses as those in Lemma 2, and suppose moreover that there exist $M_1<0<M_2$, both finite, such that*

$M_1 \leqslant x(t) \leqslant M_2$, for $t \geqslant -1$, and $(\partial f/\partial y)(x, y) \neq 0$ for each (x, y) such that $M_1 \leqslant x, y \leqslant M_2$. Then, the sequence (or the finite family, if there are only a finite number of zeros) $\#(x_{t_n})$ is non-increasing. Finally, $\#(x_t) \leqslant \#(\varphi)+1$.

Proof. For this proof it is convenient to write Eq. (2) in the form

$$x'(t) = a(t)\, x(t) + b(t)\, x(t-1), \tag{6}$$

where $a(t)$ and $b(t)$ are obtained by using the Taylor expansion of zeroth order of $f(x, y)$ near $(0, 0)$, and then

$$y'(t) = c(t)\, y(t-1), \tag{7}$$

where $y(t) = \exp(-\int_0^t a(s)\, ds)\, x(t)$, and $c(t) = b(t) \exp(\int_{t-1}^t a(s)\, ds)$. We note that x and y have the same zeros, and the assumption on $(\partial f/\partial y)(x, y) \neq 0$ is transformed into $b(t) \neq 0$, for each $t \geqslant 0$, which implies the same property for $c(t)$. Denote by t_n the sequence of zeros of x on $\mathbb{R}^+$ (which are also those of y). From $y(t_n) = y(t_{n+1}) = 0$, it follows that $y'(t)$ takes the value 0 at some point $t^* \in\,]t_n, t_{n+1}[$; therefore we have $y(t^*-1) = 0$. We saw in the proof of Lemma 2 that the count number $\#(x_t)$ is incremented by at most 1 each time a zero of x is passed from left to right. Now, let us vary t from t_n to t_{n+1}. $\#(x_t)$ may have increased from 1 at t_n but it loses 1 when t crosses the value t^*, since the point t^*-1 will no longer be counted. Therefore, $\#(x_t) \leqslant \#(x_{t_n}) - 1$, for $t^* < t < t_{n+1}$, so $\#(x_{t_{n+1}}) \leqslant \#(x_t) + 1 \leqslant \#(x_{t_n})$, for t near t_{n+1}. The proof is complete. ∎

In accordance with [5, Definition 1.2] we may define the discrete Lyapunov function for such functions φ to be $V(\varphi) = \#(x_{t_1})$, where t_1 is the first zero of the solution x on $\mathbb{R}^+$. Clearly, for each $t_n < t \leqslant t_{n+1}$, $V(x_t) = \#(x_{t_{n+1}})$. We adopt the convention that, if there are only finitely many zeros, and t_F is the last one, then $V(x_t) = 0$, for $t > t_F$. This choice differs from that made in [5], where the author assigns the value 1 when there is no zero. In this way, Cao does not distinguish between slowly oscillating solutions (a class of solutions which cancel at most once in each interval of length 1 [6]) and solutions which cease oscillating after a finite time. This is justified in Cao's perspective by the fact that the author is mainly interested in small solutions; neither of the two classes of solutions above can be a small solution. Our purpose is different from Cao's. We are motivated by the study of oscillatory properties of all possible types.

Whatever definition we take, it follows immediately from Proposition 4 that V is non-increasing along the solutions.

PROPOSITION 5. (a) *Suppose the same hypotheses as those in Lemma 2, and suppose moreover that $b \neq 0$. Let φ in $\mathscr{C}$ such that $\#(\varphi) < +\infty$. Assume* *that the solution x of (2) such that $x_0 = \varphi$ goes to zero as $t \to +\infty$. Then the function $V(x_t)$ is ultimately non-increasing. More precisely, there exists $T \geqslant 0$ such that $V(x_t) = \text{Constant}$ for $t \geqslant T$. Denote $\mathcal{N}^+ = \lim_{t \to +\infty} V(x_t)$. We also have* $\limsup_{t \to +\infty} \#(x_t) = \mathcal{N}^+$.

(b) *Suppose now that* 0 *is a hyperbolic point of Eq.* (2), *and z is a solution of* (2) *defined on* $\mathbb{R}$, *such that $z(t) \to 0$ as $t \to -\infty$. Then, $\#(z_t) < +\infty$, for each t in $\mathbb{R}$. There exists $T \leqslant 0$, such that $V(z_t) = \text{Constant}$, for $t \leqslant T$. Denote $\mathcal{N}^- = \lim_{t \to -\infty} V(z_t)$. We also have* $\limsup_{t \to -\infty} \#(z_t) = \mathcal{N}^-$.

(c) *Suppose finally that* 0 *is hyperbolic as in* (b). *Let $z \neq 0$ be a solution of* (2) *on* $\mathbb{R}$, *such that $z(t)$ approaches* 0 *as $t \to +\infty$, and $\partial f/\partial y \neq 0$ at each point of a square $[M_1, M_2] \times [M_1, M_2]$, where $M_1 \leqslant z(t) \leqslant M_2$, for every t in* $\mathbb{R}$. *Then, we have $\mathcal{N}^+ \leqslant \mathcal{N}^-$.*

Proof. For part (a), we observe first that, in view of Lemma 2, $\#(x_t) < +\infty$ for each $t \geqslant 0$. Now, from the assumption $b \neq 0$, we deduce $M > 0$, such that $\partial f/\partial y \neq 0$ at each point (x, y), for which $|x|$, $|y| \leqslant M$. Moreover, there exists τ, such that for $t \geqslant \tau - 1$, $|x(t)| \leqslant M$. We are in a position to apply Proposition 4 with $-M_1 = M_2 = M$, and to conclude that $V(x_t)$ is non-increasing for $t \geqslant T$. Since V takes only integer values and is non-negative, we may conclude that $V(x_t)$ is constant on an interval $[T, +\infty[$. Now if t_n denotes the sequence of successive zeros of x, we have, for n large enough and $t_n < t < t_{n+1}$, $\#(x_t) \leqslant \#(x_{t_n})$. So assuming that n is large enough for $t_n \geqslant T$, we have $\#(x_t) \leqslant \mathcal{N}^+$ for $t > t_n$, which yields the last result of part (a).

Let us turn to part (b). z_t has a principal part at $-\infty$ equivalent to a solution of the linear equation (3), which is a combination of exponential solutions with a positive real part exponent, $\#(z_t)$, which asymptotically at $-\infty$ is the same as the corresponding number for the principal part of z_t at $-\infty$, is therefore not larger than $I^+(a, b)/\pi$. So, $\#(z_t)$ is uniformly bounded on $\mathbb{R}$. Choose $\tau \leqslant 0$, so that $|z(t)| \leqslant M$, for $t \leqslant \tau$. Then, Proposition 4 applies on $]-\infty, \tau]$ and yields that $V(z_t)$ is non-increasing on this interval. Since it is bounded above by the number $I^+(a, b)/\pi$, we conclude that $V(z_t)$ keeps a constant value for $t \leqslant T$ for some finite T. This constant value is denoted $\mathcal{N}^-$. There are two situations: either $\mathcal{N}^- = 0$, which means that $z(t) \neq 0$ for each $t \leqslant T$ (in this case, the last result of part (b) follows trivially), or $\mathcal{N}^- \geqslant 1$. In the latter case, the zeros of z on $]-\infty, T]$ constitute an unbounded decreasing sequence t'_n, and we have $N^- = \#(z_{t'_n})$, for each n. For $t'_{n+1} < t < t'_n$, we have $\#(z_t) \leqslant \#(z_{t'_{n+1}}) = \mathcal{N}'$, completing the proof of part (b).

Part (c) follows immediately from (a) and (b) after it is noted that the assumption made now leads to the conclusion that $V(z_t)$ is non-increasing on the entire real line. ∎

We may now restate [5, Theorem 4.1] in the case of interest to us here, that is, equations of type (2) generating a monotone semi-flow. Our assumptions differ slightly from those in [5, Theorem 4.1] but the proof follows the same lines; therefore we omit it. It may be of interest to point out that the same idea as that used in [5, Theorem 4.1] has been considered, and indeed applied in a very special case, more than ten years earlier in [1, 2]. However, it was then impossible to reach the generality achieved in [5] because the essential facts contained in [5, Theorems 1.5, 2.2] were not known.

THEOREM 6. *Assume that* f *is* C^1, $f(0, 0) = 0$, $b = (\partial f/\partial y)(0, 0) > 0$. *Select* $M_1 < 0 < M_2$ *both finite such that* $\partial f/\partial y > 0$ *at each point of the product* $[M_1, M_2] \times [M_1, M_2]$. *Then, no homoclinic orbit through* $x = 0$ *may exist with range within the interval* $[M_1, M_2]$.

COROLLARY 7. *Assume that* f *is as in Theorem* 6 *and verifies any one of the following additional assumptions*:

(I) $(\partial f/\partial y)(x, y) > 0$ *for each* $(x, y) \in \mathbb{R}^2$;

(II) $(\partial f/\partial y)(x, y) > 0$ *for each* $(x, y) \in]z_1, +\infty[$ (*resp.* $]-\infty, z_2[$), *where* z_1 (*resp.* z_2) *is a steady state of Eq.* (2), $z_1 < 0$ (*resp.* $z_2 > 0$);

(III) $(\partial f/\partial y)(x, y) > 0$ *for each* (x, y), $z_1 < x, y < z_2$, *where* $z_1 < 0 < z_2$ *are two steady states of Eq.* (2).

Then, no homoclinic orbit of (2) *through* 0 *can exist.*

Proof. The only thing we have to prove is that for each possible homoclinic orbit z of (2), there exists $M_1 < 0 < M_2$, both finite, such that $M_1 \leqslant z(t) \leqslant M_2$, for each $t \in \mathbb{R}$, and $(\partial f/\partial y)(x, y) > 0$ for each (x, y) in the product $[M_1, M_2] \times [M_1, M_2]$. This result is obvious in the case (I). We prove it in the case (II) with a steady state $z_1 < 0$. The proofs of the other cases follow the same lines. All we must prove is that z does not take the value z_1, therefore, $z(t) > z_1$, for all t. Since z goes to 0 at $\pm\infty$, assume that $\liminf z \leqslant z_1$ implies that for some t_0, $z(t_0) = z_1$.

We can choose for t_0 the first point, starting from $-\infty$, to have this property. So, $z(t) > z_1$ for $t < t_0$. On the same interval, Eq. (2) leads to the inequality

$$z'(t) \geqslant f(z(t), z_1),$$

which can be written

$$z'(t) \geqslant a(t)(z(t) - z_1),$$

where $a(t) = \int_0^1 (\partial f/\partial x)(z_1 + \theta(z(t) - z_1), z_1)\, d\theta$ is continuous. Integrating the latter inequality from t to t_0, $t < t_0$, we obtain $z(t) \leqslant z_1$, for $t < t_0$, in contradiction with the definition of t_0. We may take $M_1 = z_1$, $M_2 = \max\{z(t) : t \in \mathbb{R}\}$. ∎

Remark 8. Equation (1) is an example of situation (II).

THEOREM 9. *Suppose f is C^1, $f(0, 0) = 0$, and f verifies any one of the assumptions stated in Corollary 7 (which in particular means that $b > 0$). Suppose moreover that the characteristic equation associated with the linearzation of Eq. (2) near 0 has no imaginary root and has at least two roots with a positive real part. Finally, assume that each solution of (2) exists for all positive time. Then, Eq. (2) has at least one sustained oscillatory solution.*

Proof. In view of [1, Theorem 3.2.5] or [2], the conditions of Theorem 9 entail that if Eq. (2) has no homoclinic orbit, then they have at least one sustained oscillatory solution. Therefore, the conclusion follows Corollary 7 readily. ∎

In fact, more can be said about the number and the construction of such solutions, combining the monotonicity property (see the Introduction) of the semi-flow due to the fact that $f(x, y)$ is increasing in y, and the discrete Lyapunov property studied by Yulin Cao. We would like to elaborate on that now. Let us first recall a very simple procedure (explained notably in the proof of [1, Theorem 2.2.1]) that we employed in order to construct oscillatory solutions.

Select two arbitrary functions $\varphi_0, \varphi_1, \varphi_0 < 0 < \varphi_1$, with φ_0 and φ_1 linearly independent. Then, consider the segment of data

$$\varphi_\lambda = (1 - \lambda)\, \varphi_0 + \lambda \varphi_1, \qquad 0 \leqslant \lambda \leqslant 1.$$

It is ordered through the usual order on the space of continuous functions on $[-1, 0]$. If we denote by x_λ the solutions of (2) associated with φ_λ, we have

$$\lambda < \mu \Rightarrow x_\lambda \leqslant x_\mu.$$

We then consider the sets $\Lambda^\pm$ defined by

$$\Lambda^\pm = \{\lambda \in [0, 1] : \pm x_\lambda(t) > 0, \text{for all large } t\}.$$

From the simple fact that oscillatory solutions must cancel at least once on each delay interval, we may conclude that Λ^+ and Λ^- are open subsets of $[0, 1]$ and are disjoint, and $0 \in \Lambda^-$, $1 \in \Lambda^+$. Therefore, the union $\Lambda^+ \cup \Lambda^- \neq [0, 1]$, which implies that there are solutions x_λ which oscillate over their entire domain.

Using the above procedure and the discrete Lyapunov function, we can prove the following.

PROPOSITION 10. *Under the same assumptions as those in Theorem 9, Eq. (2) has oscillatory solutions x with the property that* $V(x_t) = 2$, *for each* $t \geqslant 0$. *For such solutions, we also have* $\liminf_{t \to +\infty} \#(x_t) = 1$.

More generally, for every oscillatory solution x such that $\#(x_s) = 1$ *for some s, it is true that* $V(x_t) = 2$ *for* $t \geqslant s$ *and* $\liminf_{t \to +\infty} \#(x_t) = 1$.

Proof. We need only choose φ_0 and φ_1 in the procedure explained above in such a way that any convex combination of these two functions has at most one zero on $[-1, 0]$. A sufficient condition for this is that the function $\varphi_1(\theta)/\varphi_0(\theta)$ be strictly monotone on $[-1, 0]$.

Select a function $\zeta = \varphi_\lambda$ which leads to an oscillatory solution. Denote by t_0 the zero of ζ. It is not difficult to see that $t_0 \in]-1, 0[$. The next zero of the solution $z = x_\lambda$ is in $]0, 1[$. Denote it t_1.

We have $\#(z_{t_1}) \leqslant 2$. Assuming that $\#(z_{t_1}) = 1$ leads to the conclusion that z no longer cancels after t_1. So, $\#(z_{t_1}) = 2 = V(\zeta)$. By construction, z is oscillatory, which implies that z has infinitely many zeros. On the other hand, $V(\zeta) < +\infty$, which implies that the zeros constitute an unbounded increasing sequence t_n. For each n, we have $1 \leqslant \#(z_{t_n}) \leqslant 2$, which, for the reason given before, gives $\#(z_{t_n}) = 2$. Therefore, we deduce from the definition of V that $V(z_t) = 2$, for each $t \geqslant 0$.

In order to prove the second part of Proposition 10, it is convenient to write the equation in the form (7), changing z into y, which has the same zeros as z. For each $n \geqslant 1$, there is at least one point $t'_n \in]t_n, t_{n+1}[$ such that $y'(t'_n) = 0$. This implies $y(t'_n - 1) = 0$, from which we may conclude that $\#(y_t) = 1$ for $t'_n < t < t_{n+1}$, and then $\liminf_{t \to +\infty} \#(y_t) = 1$.

For the last part of Proposition 10, we need only observe that we can repeat the above proof with $\psi = x_s$ and $z(t) = x(t+s)$. ∎

Proposition 10 applies in particular to the linear equation, obtained by linearization of (2) near $x = 0$, that is, Eq. (3). In view of the proof of Proposition 1 and the fact that solutions of linear equations are asymptotically equivalent to a combination of exponential solutions, we then deduce from Proposition 10 that the lowest positive imaginary part of roots of the characteristic equation associated with (3) lies in the interval $]\pi, 3\pi/2[$.

THEOREM 11. *Suppose f verifies the same conditions as those in Theorem 9. Then, each oscillatory solution x such that* $\#(x_s) = 1$ *for some s has sustained oscillations.*

Proof. The characteristic equation associated with Eq. (3) has one real root, which by standard results on monotone semiflows [7] yields the leading eigenvalue, the eigenvalue with the largest real part. All roots have multiplicity one. So, assuming that the characteristic equation has at least two roots with a positive real part implies that it has a complex root with a positive real part. In other words, $I^+(a, b) > 0$. This also implies that the oscillatory solutions of the linear equation which go to zero at $+\infty$ have more than one zero on each unit interval. In fact, $I^-(a, b) \geqslant 3\pi$, which means that the distance between two consecutive zeros of a function of type $\exp(\alpha t)\cos(\beta t + \gamma)$, with $\alpha < 0$, is not larger than $\frac{1}{3}$. Such a function has at least two zeros in each interval of length 1. Now, suppose x is an oscillatory solution of (2) and $\#(x_s) = 1$ for some s and x goes to 0 as $t \to +\infty$. From Proposition 10, we deduce that $V(x_t) = 2$ for $t \geqslant s$, which in particular means that x has only a finite number of zeros on each unit interval. Theorem 2.2 of [5] then applies and guarantees that x will not go to 0 faster than any exponential, therefore has a principal part at $+\infty$ which is a combination of exponential solutions with a negative real part exponent. This implies that for t large enough $\#(x_t) \geqslant 2$, which contradicts the property established in Proposition 10 that $\liminf_{t \to +\infty} \#(x_t) = 1$. This contradiction completes the proof of Theorem 11. ∎

The above procedure shows that as soon as the characteristic equation has more than one root with a positive real part, it has infinitely many sustained oscillatory solutions. In fact, most oscillatory solutions are then sustained. This fits the idea which motivated our research, that for monotone delay equations, the appearance of unstable oscillations in the linear equation probably reflects the appearance of sustained oscillations in the non-linear equation.

Suppose now that x has sustained oscillations and is uniformly bounded on a positive semi-axis: this, by the way, is the case for all oscillatory solutions of our model equation. Then, the points lying in the omega-limit set of x_t give rise to sustained oscillatory solutions defined on the entire real axis. It would be interesting to look at such solutions and determine what kind of functions they are.

2. Comments on Yulin Cao's Proof of [5, Theorem 2.2]

Theorem 2.2 of [5] states that a solution of (2), if it is finitely oscillatory (i.e., $V < +\infty$), decays at most exponentially at $+\infty$. In our opinion, this result is a breakthrough in the study of delay differential equations and will probably have consequences other than the one we derived in Section 1.

Until now, it has been known only for time-independent linear delay differential equations [6] and is known to be false in the case of time-dependent delay equations [6]. A related though weaker result was proved in [4] for monotone delay differential systems: solutions whose components cease to oscillate eventually cannot go to 0 at a super-exponential rate. Surprisingly, the proof of Theorem 2.2 is based on quite an elementary although very neat observation, which is, essentially, that if you assume a solution has a rapid decay from t to $t+1$, it indicates rapid oscillations before t. It seems to us however that the proof is a little difficult to follow; part of the construction can be put aside. We propose another proof based on the same idea but, we hope, concentrating on the main mechanism of this result. Before we start, we would like to mention that the converse of this result is more or less known: at least when 0 is a hyperbolic steady state, solutions which decay at most exponentially are asymptotically equivalent to a combination of exponential solutions of the linear equation, therefore are finitely oscillatory.

So let x be a solution of Eq. (2), where we only assume that f is C^1, $f(0,0)=0$, and $b=(\partial f/\partial y)(0,0)\neq 0$. Suppose x is defined on $\mathbb{R}^+$, and for some t_0, $\#(x_{t_0})<+\infty$, and $x(t)\to 0$ as $t\to+\infty$. We may choose t_0 large enough for Proposition 4 to be applicable on $[t_0, +\infty[$. Therefore, we have $\#(x_t)\leqslant \#(x_{t_0})+1$, for $t\geqslant t_0$.

Set $N=\#(x_{t_0})+1$. We want to prove that x does not go to zero faster than any exponential. The proof is done by contradiction. First of all, the equation is rewritten in the form of Eq. (7), and x is changed to y. y has the same zeros as x and goes to 0 faster than any exponential, too. $c(t)\to b\exp(-a)=_{\text{def}} c$, as $t\to+\infty$. $c(t)$ is defined in (7). A first preparatory result is [5, Lemma 2.3]: For each $T>0$, $0<\delta<1$, there exists a constant $C=C(T,\delta)$, such that for each interval $I\subseteq[-T,0]$, $|I|=\delta$, $\min_{s\in I}|y(s)|\leqslant C(T,\delta)\max_{0\leqslant s\leqslant 1}|y(s)|$. Let us now express the fact that $y\to 0$ faster than any exponential. This implies notably that for a sequence $t_n\to+\infty$, we have

$$\|y_{t_n+1}\|/\|y_{t_n}\|\to 0, \qquad \text{as} \quad n\to+\infty.$$

In order to simplify the notations, we denote generically any pair of the family $\mathscr{F}$ by

$$\mathscr{F}=\{(c(t+t_n),\ y(t+t_n)/\|y_{t_n}\|):n\in\mathbb{N}\}.$$

The notation for a generic pair is (d, z). For each pair, we have the relation

$$z'(t)=d(t)\,z(t-1) \qquad (t\geqslant -\tau),$$

for some τ.

We now summarize the main property of the family $\mathscr{F}$ due to the assumptions placed on y that we use throughout the rest of the proof.

(*) For each $T>0$, $\varepsilon>0$, we can find $(d, z)\in\mathscr{F}$, such that $|d(t)-c|\leqslant\varepsilon$, for $t\geqslant -T$, and $\|z_1\|\leqslant\varepsilon$, while of course $\|z_0\|=1$. Finally, $\#(z_t)\leqslant N$ for $t\geqslant -T$.

A contradiction is obtained by showing that $\#(z_t)$ is actually larger than N, for some $t\geqslant -T$.

For a moment, fix $T>0$, $0<\delta<1$, and $\varepsilon>0$. Suitable values of these parameters are determined later. Choose z as above. For each interval I, of length δ, $I\subseteq[-T, 0]$, we have

$$\min\{|z(s)| : s\in I\}\leqslant C\varepsilon,$$

where $C=\max(C(T,\delta), 1)$. We take ε small enough for $C\varepsilon<1$.

Now choose a point $\theta_0\in[-1, 0]$, where $|z_0|$ is maximal: $|z(\theta_0)|=1$. On each interval $[\theta_0-\delta, \theta_0]$, $[\theta_0, \min(\theta_0+\delta, 0)]$, the function $|z|$ also takes values not larger than $C\varepsilon$. Therefore, $|z|$ has a local maximum at some point $\theta_0'\in[\theta_0-\delta, \theta_0]$, which implies that $z'(\theta_0')=0$. In view of the equation verified by z, we obtain $z(\theta_0'-1)=0$.

At the same time, we may note that on each side of θ_0, there are points, say t_1, $\theta_0-\delta<t_1<\theta_0$, and t_2, $\theta_0<t_2<\theta_0+\delta$, where

$$|z'(t_i)|\geqslant(1-C\varepsilon)/\delta \qquad (i=1, 2).$$

We may choose $t_1<\theta_0'$. In fact:

Either $\min\{|z(t)| : \theta_0'\leqslant t\leqslant\theta_0\}<1$. In this case, $|z(\theta_0)|$ is a local maximum. We can take $\theta_0'=\theta_0$, which yields the claimed fact immediately.

Or $\min\{|z(t)| : \theta_0'\leqslant t\leqslant\theta_0\}\geqslant 1$. In this case, $|z|$ drops from not less than 1 to below $C\varepsilon$ on the interval $[\theta_0-\delta, \theta_0']$. So, there exists $t_1\in[\theta_0-\delta, \theta_0']$ for which the desired inequality holds.

Set $M=\max\{d(s) : s\geqslant -T\}$. We then have $|z(t_i-1)|\geqslant(1-C\varepsilon)/\delta M$, for $i=1, 2$. Since T is still arbitrary, we now choose δ and ε so that $(1-C\varepsilon)/\delta M\geqslant 1$. First, choose δ so that $\delta M<1$. Next, choose ε small enough for the ratio to be not less than 1.

Summarizing what we have obtained so far, we find that on $[-2-\delta, -1+\delta]$, the function z has a zero (the point $\theta_0'-1$) surrounded by two points where $|z|$ is "large," namely, the points t_1-1 and t_2-1. Precisely, we have

$$\theta_0-\delta-1\leqslant t_1-1\leqslant\theta_0'-1\leqslant t_2-1\leqslant\theta_0+\delta-1,$$

$z(\theta_0'-1)=0$, $|z(t_i-1)|\geqslant 1$, $i=1, 2$.

Using the same idea as that above, we can now produce a maximum for $|z|$ on $]\theta_0'-1, t_2-1+\delta]$ and one on $[t_1-1-\delta, \theta_0'-1[$, each of which yields a zero for z on the interval $[\theta_0-2-2\delta, \theta_0-2+2\delta]$.

The same procedure can be repeated. Suppose that at step $k+1$, we have k zeros $\theta_1, \theta_2, ..., \theta_k$, all contained in $[\theta_0 - k - 2(k-1)\delta, \theta_0 - k + 2(k-1)\delta]$, and two points t_1, t_2,

$$\theta_1 - \delta < t_1 < \theta_1, \qquad \theta_k < t_2 < \theta_k + \delta,$$

such that $|z(t_i)| \geqslant 1$, $i = 1, 2$.

Each interval (θ_j, θ_{j+1}), $1 \leqslant j \leqslant k-1$, contains at least one point θ'_j where $z'(\theta'_j) = 0$, yielding $z(\theta'_j - 1) = 0$. We also have an extremum for z on $[t_1 - \delta, \theta_1[$, say at θ'_0, and one on $]\theta_k, t_2 + \delta]$, say at θ'_k. Therefore, we obtain $z(\theta'_j - 1) = 0$, for $j = 0, ..., k$.

Set $\tilde{\theta}_j = \theta'_{j-1} - 1$, for $j = 1, ..., k+1$. By construction, these points lie in the interval $[\theta_0 - k - 1 - 2\delta - 2(k-1)\delta, \theta_0 - k - 1 + 2\delta + 2(k-1)\delta] = [\theta_0 - (k+1) - 2k\delta, \theta_0 - (k+1) + 2k\delta]$. Finally, from $|z(\theta'_0)| \geqslant 1$, we deduce that there exists $t'_1 \in [\theta'_0 - \delta, \theta'_0]$, where $|z'(t'_1)| \geqslant (1 - C\varepsilon)/\delta$, therefore, $|z(t'_1 - 1)| \geqslant 1$.

Set $\tilde{t}_1 = t'_1 - 1$. We have $|z(\tilde{t}_1)| \geqslant 1$, with $\tilde{\theta}_1 - \delta \leqslant \tilde{t}_1 \leqslant \tilde{\theta}_1$. Similarly, we can construct $\tilde{t}_2 \in [\tilde{\theta}_{k+1}, \tilde{\theta}_{k+1} + \delta]$. The procedure can be continued as long as the interval $[\theta_0 - k - 2(k-1)\delta, \theta_0 - k + 2(k-1)\delta]$ is contained in $[-T, 0]$, which, with $\theta_0 \in [-1, 0]$, holds if

$$-k + 2(k-1)\,\delta \leqslant 0 \qquad \text{and} \qquad -1 - k - 2(k-1)\,\delta \geqslant -T.$$

The first inequality holds with no restriction on k if we assume that $\delta \leqslant \frac{1}{2}$. The second inequality, together with $\delta \leqslant \frac{1}{2}$, holds if $k \leqslant (T-1)/2$. We are now in a position to choose T in order to obtain a contradiction.

We take $T = 2N + 3$. With this choice, the above procedure can be repeated at least until $k = N + 1$, and it yields $k = N + 1$ zeros for z on an interval of length $4N\delta$, contained in $[-T, +\infty[$. On the other hand, we know that z has at most N zeros on each interval of length 1 contained in $[-T, +\infty[$. So if we restrict δ in such a way that $4N\delta < 1$, we will arrive at a contradiction.

In order to complete the proof, let us show that we can indeed choose T, δ, and ε in such a way that all the conditions stated above hold. First, we take $T = 2N + 3$. Next, we choose δ, so that $\delta < \min(\frac{1}{2}, 1/M, 1/4N)$; then ε can be chosen so that $\varepsilon < (1 - \delta M)/C(T, \delta)$. Finally, we choose z, that is, a translate of y, $y(t + t_n)$, with n large enough for condition $(*)$ to hold.

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