Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 26, No. 2, pp. 335–341, 1996
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0362–546X/96 $9.50 + .00

0362–546X(94)00260–6

PERIODIC AND ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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(*Received* 25 *January* 1994; *received in revised form* 29 *June* 1994; *received for publication* 13 *September* 1994)

Key words and phrases: Ordinary differential equation in a Banach space, almost-periodic solution, global solution, dissipative operator, semi-inner product, strictly convex norm, measure of noncompactness.

1. INTRODUCTION

Let E be a real Banach space, with norm $|.|$. We consider the following nonlinear ordinary differential equation

$$\frac{dx}{dt} + g(x)Ax = h(t), \tag{1.1}$$

where h is periodic (respectively, almost periodic), $A \in L(E)$ is a bounded linear operator and g is a function whose properties are stated hereafter. To this equation, in which h plays the role of a forcing term, we are interested in finding periodic (resp. almost periodic) solutions. A special case of equation (1.1) was introduced by Bayliss in [1]

$$x'(t) + |x|^{\alpha}x(t) = h(t) \tag{1.2}$$

with $\alpha > 0$. Equation (1.2) was also considered by Arino and Hanebaly [2] who extended Bayliss' results from Hilbert spaces to Banach spaces, however assuming $\alpha < 1$. Related results have been obtained by Kartsatos [3, 4]. The present work is an extension of Arino and Hanebaly's results, using a completely different method. While these authors put the emphasis on the dissipativeness of (1.1) and made extensive use of the semi-inner product as a substitute to the inner product in a Hilbert space, our main effort here is on establishing compactness properties of trajectories under appropriate conditions on h. A crucial observation is that the term $g(x(t))$ may be absorbed using a change of the time variable.

Throughout the paper, we will assume the following basic hypotheses on g and h:

(A1) g: $E \to \mathbb{R}^+$, $g(x) > 0$, if $x \neq 0$.
- g is continuous and sends bounded sets into bounded sets.
- g is coercive in the sense that $g(x) \to +\infty$, as $|x| \to +\infty$.
- For each $\xi > 0$, $\inf_{|x| \geq \xi} g(x) > 0$;

(A2) • h: $\mathbb{R} \to E$, is continuous and bounded.

2. GLOBAL EXISTENCE OF SOLUTIONS

THEOREM 2.1. In addition to (A1), and (A2), we assume:

(A3) $x \to g(x)Ax$ is locally Lipschitz continuous in x, continuous in (t, x); and
(A4) $x \to -Ax$ is strongly dissipative

(i.e. for some $c > 0$, $(Ax, x)_+ \geq c|x|^2$, where $(.,.)_\pm$ denote the $\pm$ semi-inner product [3,5].

Then, the Cauchy problem for equation (1.1) is well posed, that is: For each $(t_0, x_0) \in \mathbb{R} \times E$, there exists one and only one function x defined on an interval $[t_0, t_0 + a[$ $(a > 0)$ such that: x verifies equation (1.1) on its domain, with $x(t_0) = x_0$. x can be continued as a solution of (1.1) over $[t_0, +\infty[$ in a unique way. Moreover, its extension, still denoted x, is uniformly bounded on $[t_0, +\infty[$. Finally, $|x(t)|$ is ultimately bounded by a quantity which depends only on h (not on x_0).

Proof. Local existence and uniqueness follow readily from the fact that, by (A1) and basic assumption on h, the function

$$F(t, x) = -g(x)Ax + h(t) \tag{2.1}$$

is locally Lipschitz continuous in x, continuous in (t, x).

Denote x the maximal solution. The domain of x is an interval $[t_0, T^*[$. We have to prove that $T^* = +\infty$, and x is uniformly bounded on $[t_0, +\infty[$. As a preliminary, let us observe that the function F, as defined by (2.1), is bounded on bounded subsets of E. In fact, we have $|F(t,x)| \leq |h|_\infty + \sup_{|x| \leq R} g(x).|A|_{L(E)} R$, for $|x| \leq R$. We also know, by (A1), that: $\sup_{|x| \leq R} g(x) < +\infty$. Therefore, the noncontinuation principle [6] holds, that is: if $T^* < +\infty$ then we must have: $\limsup_{t \to T^*} |x(t)| = +\infty$.

To conclude the proof on existence and boundedness, we only have to prove that x is uniformly bounded on its domain. For this, we look at

$$\frac{1}{2} \frac{\mathrm{d}^-}{\mathrm{d}t}(|x(t)|^2).$$

It is well known [5] or [6] that

$$\frac{1}{2} \frac{\mathrm{d}^-}{\mathrm{d}t}(|x(t)|^2) = (x'(t), x(t))_-.$$

Using standard properties of the semi-inner product, we arrive at

$$(x', x)_- \leq (-g(x)Ax, x)_- + (h, x)_+$$

$$(x', x)_- \leq -g(x)\,(Ax, x)_+ + |h|_\infty\,|x(t)|$$

So, in view of (A4), we may deduce the following inequality

$$|x(t)| \frac{\mathrm{d}^-}{\mathrm{d}t}(|x(t)|) \leq -g(x).c.|x(t)|^2 + |h|_\infty |x(t)|,$$

$$\frac{\mathrm{d}^-}{\mathrm{d}t}(|x(t)|) \leq -g(x).c.|x(t)| + |h|_\infty. \tag{2.2}$$

From the coerciveness of g, it follows that the right-hand side of (2.2) is negative if $|x(t)|$ is large enough. For each $\xi > 0$, let us introduce the number $R_{h,\xi}$ (finite, due to coerciveness)

$$R_{h,\xi} = \inf\{\rho > 0 : -g(x)c|x| + |h|_\infty < -\xi, \qquad \text{for } |x| \geq \rho\}.$$

Then, we have: $|x(t)| \leq \max(R_{h,\xi}, |x_0|)$, for $t < T^*$. We will now prove that: $\limsup_{t \to +\infty} |x(t)| \leq R_{h,\xi}$.

In fact, assume on the contrary, that $\limsup |x(t)| > R_{h,\xi}$. For each R' such that, $R_{h,\xi} < R' < \limsup |x(t)|$, we have $|x(t)| > R'$, for each t. Otherwise, if $|x(t')| < R'$ for some t' then, $|x(t)| \leq R'$ for each each $t > t'$, since, the right-hand side of (2.2) is negative for $|x| > R_{h,\xi}$. Therefore, $\limsup_{t \to +\infty} |x(t)| \leq R'$, a contradiction. This yields $|x(t| \geq \limsup_{s \to +\infty} |x(s)|$ for each t, from which it thus follows that $\liminf_{s \to +\infty} |x(s)| \geq \limsup_{s \to +\infty} |x(s)|$, that is: $|x(t)|$ converges to a limit. However, then there exists t_0 such that

$$t \geq t_0, \qquad |x(t)| \geq R_{h,\xi},$$

therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(|x(t)|) \leq -\xi, \qquad \text{for } t \geq t_0.$$

The latter inequality leads to an obvious contradiction. The proof is complete. ■

3. PRECOMPACTNESS OF POSITIVE ORBITS

THEOREM 3.1. Assume (A1) through (A4) hold, and that the forcing term h has a relatively compact range (i.e. clos$\{h(\mathbb{R})\}$ is compact). Let $(t_0, x_0) \in \mathbb{R} \times E$. Denote x the solution of (1.1), such that: $x(t_0) = x_0$. Then, $x([t_0, +\infty[)$ is relatively compact.

The proof of theorem 3.1 makes use of the measure on noncompactness for which we refer to [5] or [6].

Proof. We consider two situations:

(1) $\liminf_{t \to +\infty} |x(t)| > 0$. In this case, there exist $t_1 \geq t_0$, and $\xi > 0$ such that $|x(t)| \geq \xi$, for $t \geq t_1$.

Therefore, in view of (A1) we can find $\eta > 0$ such that $g(x(t)) \geq \eta$, for $t \geq t_1$. So, we may deduce, using (A1) and an estimate of $|x(t)|$ given in the proof of theorem 2.1, that there exists $0 < a < b < +\infty$, with, $a \leq g(x(t)) \leq b$, for $t \geq t$.

$$a = a(\xi) = \inf\{|g(x)| : |x| \geq \xi\}$$
$$b = b(h, x_0) = \sup\{|g(x)| : |x| \leq \max(R_h, |x_0|)\}$$
$$\text{with } R_h = \inf\{\rho : -cg(x)|x| + |h|_\infty \leq 0, |x| \geq \rho\}.$$

Introduce the new time variable

$$\tau(t) = \int_{t_1}^{t} g(x(s))\,\mathrm{d}s.$$

Denote $y(\tau) = x(t(\tau))$, $x(t) = y(\tau(t))$.

In terms of y and τ, equation (1.1) reads as

$$y'(\tau)+Ay(\tau)=\frac{h(t(\tau))}{g(x(t(\tau))}=\hat{h}(\tau). \tag{3.1}$$

Now, if we denote $K=\overline{\mathrm{co}}(h(\mathbb{R})$, by assumption of theorem 3.1, $h(\mathbb{R})$ has a compact closure and so, using for example a property of the measure noncompactness, K is compact too. In terms of K, we can express the range of $\hat{h}$. In fact, we have

$$\hat{h}(\mathbb{R}^+)\subset\{rk:b^{-1}\le r\le a^{-1},\qquad k\in k\}=\hat{K}_{a,b}.$$

$\hat{K}_{a,b}$ is clearly a compact, convex subset of E.

Using the following representation of y

$$y(\tau)=\exp(-A\tau).y_0+\int_0^\tau \exp(-As).\hat{h}(\tau-s)\,\mathrm{d}s,$$

we will now conclude that $y(\tau)$ takes its values in some compact subset of E. From assumption (A4), we deduce that $\exp(-A+cI)t$ is a contraction for each $t\ge 0$.

For each subset X of E, and each interval J of $\mathbb{R}$, we introduce the following notation

$$K_X/J=\{\exp(-A+cI)t.x:\qquad t\in J,\qquad x\in X\},\qquad (K_X \text{ if } J=[0,+\infty[).$$

We can see that if X is compact, K_X is relatively compact too.

The fact that $\exp(-A+cI)t$ is a contraction is crucial in proving compactness of K_X. In fact, it guarantees that $\exp(-A+cI)t.x$ converges (uniformly in $x\in X$) to zero as $t\to+\infty$. Therefore, we can express

$$K_X=K_X/[0,T]\cup K_X/[T,+\infty[.$$

$K_X/[0,T]$ is compact as the continuous image of a compact set, and the diameter of $K_X/[T,+\infty[$ goes to zero as $T\to+\infty$.

Denote: $H_1=\overline{\mathrm{co}}(0,K_{\{y_0\}})$; $H_2=\overline{\mathrm{co}}(0,\overline{\mathrm{co}}(K_{\hat{K}}))$. H_1 and H_2 are compact.

From the variation of constants formula, we can see that

$$y(\tau)\in\left\{H_1+\frac{1}{\gamma}H_2\right\},\qquad \tau\ge 0.$$

The sum of two compact sets being compact, we conclude that $y(\mathbb{R}_+)$ is relatively compact. So, the same holds for the solution x for $t\ge t_1$. Thus, $x([t_0,t_1])$ being compact, the conclusion of theorem 3.1 is reached in this case.

(2) $\liminf_{t\to+\infty}|x(t)|=0$. If, moreover, $\limsup_{t\to+\infty}|x(t)|=0$, this means that $|x(t)|\to 0$, as $t\to+\infty$. Obviously in this case, $x([t_0,+\infty[)$ is relatively compact. So, let us assume that $\limsup_{t\to+\infty}|x(t)|=d>0$.

For each $0<\rho<d$, we will evaluate the measure of noncompactness of the part of x outside of the ball $B(0,\rho)$. Select a number $t_1\ge t_0$, such that $|x(t_1)|<\rho$. For each $t>t_1$, such that $|x(t)|>\rho$, denote $\bar{t}$, the largest time in $[t_1,t]$ for which $|x(\bar{t})|=\rho$. So, we have: $|x(s)|>\rho$, $\bar{t}<s<t$. Using the same argument as in the first case, we arrive at the conclusion that

$$x(s)\in\left\{H_1^\rho+\frac{1}{\gamma}H_2^\rho\right\},$$

where $H_1^\rho = K_{\overline{B}(0,\rho)} \subset \overline{B}(0, \rho)$, (because $\exp(-A + (1/\gamma)I)t$ is a contraction for each $t \geq 0$); and

$$H_2^\rho = \overline{\mathrm{co}}\big(0, \overline{\mathrm{co}}(K_{\hat{K}(a(\rho),b)}\big).$$

This implies that

$$\alpha\{x(s): |x(s)| > \rho\} \leq \alpha\left(H_1^\rho + \frac{1}{\gamma} H_2^\rho\right).$$

We already noticed that H_2^ρ is compact, so, $\alpha(H_2^\rho) = 0$. On the other hand, from $H_1^\rho \subset \overline{B}(0, \rho)$, we deduce that $\alpha(H_1^\rho) \leq C_1 \rho$ (where the constant C_1 depends the particular measure of noncompactness we are dealing with).

So

$$\alpha\{x(t): |x(t)| > \rho\} \leq C_1 \rho.$$

Now, we have

$$\alpha\{x(t): |x(t)| \leq \rho\} \leq C_2 \rho.$$

Therefore, using another property of the measure of noncompactness, we obtain

$$\alpha\{x(t): t \geq t_0\} \leq \max(C_1, C_2)\rho.$$

This being true for each $\rho > 0$, we may conclude that

$$\alpha\{x(t): t \geq t_0\} = 0. \quad \blacksquare$$

COROLLARY 3.1. Suppose the norm on E is strictly convex, and (A1) through (A4) hold. Then, for each h, continuous and almost periodic, equation (1.1) has at least one almost periodic solution.

Proof. We know by theorem 3.1 that for each $(t_0, x_0) \in \mathbb{R} \times E$, the equation (1.1) has at least one solution x defined on interval $[t_0, +\infty[$ such that $x([t_0, +\infty[)$ is relatively compact.

Then the conditions of the theorem 1.2, Chapter VII in [7] are satisfied. We may conclude that equation (1.1) has at least one almost periodic solution.

COROLLARY 3.2. Let us consider equation (1.2). In the case the norm on E is strictly convex, $\alpha \geq 0$ and h is continuous, almost periodic, then, equation (1.2) has at least one almost periodic solution.

This result is an immediate consequence of corollary 3.1. It should be noted that this statement complements the one given in [2] where the method used seems to be restricted to $\alpha < 1$.

4. CONSTRUCTION OF SOLUTIONS DEFINED ON $\mathbb{R}$, WITH A RELATIVELY COMPACT RANGE

In the case of an arbitrary Banach space, we do not know whether the conclusions of corollary 3.1 hold. In this section, we will show that at least a very partial property can be verified: namely, that there are solutions defined on $\mathbb{R}$ whose range is relatively compact. As a remark we may note that if, in addition to $h(\mathbb{R})$ relatively compact, we assume that for some sequence $(t_n)_{n\in\mathbb{R}}$, $t_n \to +\infty$, $h(t+t_n) \to h(t)$, locally uniformly with respect to t, then equation (1.1) has at least one solution y defined on $\mathbb{R}$, such that y has a relatively compact range.

In fact, we know by theorem 3.1 that if for each $(t_0, x_0) \in \mathbb{R} \times E$, we denote by x the solution of (1.1) with $x(t_0) = x_0$, then $x([t_0, +\infty[)$ is relatively compact.

Select one such solution, consider the sequence $x(t+t_n)$. For each $R > 0$, if we restrict t to vary in $[-R, R]$, it is easily seen that the sequence of functions $x(.+t_n)$ satisfies the conditions of the Ascoli–Arzela theorem [5], therefore, it has a convergent subsequence. Using the diagonal procedure, it is possible to find a subsequence which converges on each interval $[-R,R]$. Denote y the limit. Clearly, y is a solution of (1.1) defined on $\mathbb{R}$, and $\operatorname{clos}(y(\mathbb{R}) \subset \operatorname{clos}(x([t_0, +\infty[))$, therefore, y has a relatively compact range. The condition we used on h is obviously verified by almost periodic functions. However, it is a far more general assumption. However, we will now see that the same arguments as in the proof of theorem 3.1 yield the above result without any further assumption on h apart from $h(\mathbb{R})$ being relatively compact.

THEOREM 4.1. Under the same assumptions as in theorem 3.1, equation (1.1) has at least one solution x, defined on $\mathbb{R}$ and such that the range of x is relatively compact.

Proof. We introduce the following sequence of functions x_n: for each n, x_n is defined on $[-n, +\infty[$, and is a solution of (1.1) on this interval, with $x(-n) = 0$. For each $\rho > 0$, we know (see proof of theorem 3.1) that there exists a compact set K_ρ (depending only on h) such that

$$x_n(t) \in \bar{B}(0, \rho) + K_\rho, \qquad t \geq -n$$

and, we have: $\alpha(\bar{B}(0, \rho) + K_\rho) = C\rho$.

So, for each $t \in \mathbb{R}$, and $\rho > 0$,

$$\alpha(\{x_n(t) : n \geq -t\}) \leq C\rho, \qquad \text{thus,}$$

$$\alpha(\{x_n(t) : n \geq -t\}) = 0.$$

Since the x_ns are uniformly bounded too, we deduce from the Ascoli–Arzela theorem that the restriction of x_n to any bounded interval is relatively compact.

So, using the diagonal procedure, we can construct a subsequence which converges uniformly on each bounded subset of the real line towards a function x. Obviously, x is a solution of (1.1) on $\mathbb{R}$. Moreover, for each $\rho > 0$, we have $x(t) \in \bar{B}(0, \rho) + K_\rho$, for each $\rho > 0$, which leads to

$$\alpha\{x(t) : t \in \mathbb{R}\} = 0.$$

The proof of theorem 4.1 is complete. ■

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