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More on Ordinary Differential Equations Which Yield Periodic Solutions of Delay Differential Equations*

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We construct a Poincaré operator for the system

$$\frac{\mathbf{dx}}{\mathbf{dt}} = -\lambda \mathbf{x} - \mathbf{F}(\mathbf{x}), \tag{0.1}$$

where λ is a real parameter, $\mathbf{x} \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)$,

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f(x_2) \\ f(x_3) \\ f(x_1) \end{pmatrix},$$

and f is an odd C^2 function such that $f'(0) = 1$, $xf(x) > 0$, for $x \neq 0$. We also consider the case where f is C^1. We will express F in linearized form, that is,

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{G}(\mathbf{x}),$$

where $\mathbf{A}$ is the linearized part of $\mathbf{F}$ around zero and $\mathbf{G}(\mathbf{x}) = o(|\mathbf{x}|)$ near zero. Fixed points of the Poincaré operator correspond to periodic solutions of the functional differential equation

$$\frac{dx}{dt} = -\lambda x(t) - f(x(t - T/3)), \tag{0.2}$$

where T is the period of x.

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Introduction and Main Results

Our purpose in this paper is to relate the ordinary differential system (0.1) to the scalar delay differential equation (0.2), in the search of periodic solutions. This idea was initiated with great success by J. L. Kaplan and J. A. Yorke [5] for the case of the logistic equation

$$\dot{x}(t) = -\lambda f(x(t-1)). \tag{0.3}$$

They proved that under the extra assumption that f is odd, there is a continuum of periodic solutions of (0.3) which are in fact solutions of an ordinary system associated to (0.3), namely,

$$\begin{cases} \dot{x}(t) = -\lambda f(y(t)) \\ \dot{y}(t) = \lambda f(x(t)). \end{cases} \tag{0.4}$$

These are slowly oscillating periodic solutions of (0.3) with period 4 which have some symmetry property,

$$x(t+2) = -x(t).$$

By the "slowly oscillating solution" of a delay differential equation, one means classically a solution x, such that there exists an increasing sequence (t_j), $t_j \in \mathbb{R}$, $t_{j+1} - t_j$ being not less than the maximum delay such that $x(t_j) = 0$, x keeps a constant sign on $]t_j, t_{j+1}[$, and changes signs from an interval to the next one.

It is E. M. Wright [7] who first pointed out the property of slow oscillation for Eq. (0.3). K. P. Hadeler and J. Tomiuk [3] extended the analysis made on the logistic equation (0.3) to equations of type (0.2) which cannot be transformed to type (0.3).

As for (0.3), it is possible to associate to Eq. (0.2), ordinary differential systems whose periodic solutions are also solutions of (0.2).

In fact, a periodic solution of (0.2) with a period multiple of the delay is a solution of an ordinary differential system. Looking for the first integer k such that (0.2) has a periodic solution of period k, it is known that $k \geqslant 3$. It is also easily seen that there is no periodic solution x of (0.2) with period 4 such that $x(t+2) = -x(t)$. In fact assuming that x is such a solution, and denoting $y(t) = x(t-1)$, Eq. (0.2) leads to the system

$$\begin{cases} \dot{x}(t) = -\lambda x(t) - f(y(t)) \\ \dot{y}(t) = -\lambda y(t) + f(x(t)). \end{cases}$$

Letting $F(x) = \int_0^x f(u)\,du$, we have

$$\frac{d}{dt}\{F(x(t))+F(y(t))\}$$
$$=f(x(t))\cdot[-\lambda x(t)-f(y(t))]+f(y(t))[-\lambda y(t)+f(x(t))]$$
$$=-\lambda[x(t)f(x(t)+y(t)f(y(t))]<0,$$

for $\lambda>0$, $(x, y)\neq(0, 0)$. So $x\equiv 0$.

We will prove that Eq. (0.2) has periodic solutions which have some symmetry property.

The paper is organized as follows: In Section 1, we first establish a few results regarding Eq. (0.1). In particular we define a Poincaré operator adapted to the search of periodic solutions which have some symmetry property . Next, we look at the linearization of Eq. (0.1) and the Poincaré operator.

More precisely we describe the Poincaré operator corresponding to

$$\frac{\mathbf{dx}}{\mathbf{dt}}=(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}. \tag{0.5}$$

In the study of the non-linear equation, we will use the functions

$$g(\mathbf{x})=\sum_{i\neq j}\{x_j f(x_{[i+1]})+x_i f(x_{[j+1]})\},$$

where $[k]$ denotes a modulo 3 function, $[k]=k$, for $k=1, 2, 3$, and $[4]=1$, etc., and

$$\sigma(\mathbf{x})=x_1x_2+x_2x_3+x_3x_1.$$

We make the hypothesis

$$\sigma(\mathbf{x})\geqslant 0, \qquad \mathbf{x}\neq 0\Rightarrow g(\mathbf{x})>0. \tag{$*$}$$

One simple situation where $(*)$ holds is when f is monotone increasing, an assumption we will further impose in Section 3.

We will show that to each region P_i of the coordinate plane $x_i=0$, defined by the other two components of opposite signs, we can associate a Poincaré operator Π,

$$\Pi\colon \mathbb{R}^+\times\mathbb{R}^-\to\mathbb{R}^+\times\mathbb{R}^-.$$

For instance, if $\mathbf{x}^0=(x_1^0, x_2^0, 0)$ with $x_1^0>0>x_2^0$, we show that the corresponding solution meets the plane $x_2=0$ after some time $\tau>0$. If we denote the solution by $(x_1(t), x_2(t), x_3(t))$, we have $x_2(\tau)=0$, for some positive τ. Then, the Poincaré map Π is defined by

$$\Pi(x_1^0, x_2^0)=-(x_3(\tau), x_1(\tau)).$$

Section 2 is devoted to the study of periodic solutions which arise in the vicinity of zero. We extend the results of Section 1 to system (0.1). We obtain the following theorems, corresponding to the cases where f is C^2 and C^1, respectively.

THEOREM 2.1. *Consider the system*

$$\frac{\mathbf{dx}}{\mathbf{dt}} = -\lambda \mathbf{x} - \mathbf{F}(\mathbf{x}), \tag{0.1}$$

where λ *is a real parameter* $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)$,

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \mathbf{f}(x_2) \\ \mathbf{f}(x_3) \\ \mathbf{f}(x_1) \end{pmatrix},$$

where $\mathbf{f}$ *is an odd* $\mathbf{C}^2$ *function such that* $\mathbf{f}'(0) = 1$, $x\mathbf{f}(x) > 0$, *for* $x \neq 0$, *and* $(*)$ *holds. Let* $\mathbf{x}^0 = (x_1^0, x_2^0, 0)$ *be an initial value for system* (0.1) *such that* $x_1^0 > 0 > x_2^0$. *Set* $x_1^0 = \rho\mu^0$ *and* $x_2^0 = -\rho$.

Then

(i) *for every* ρ *close to zero, there exists one and only one point of the form* $(\lambda(\rho), \mu^0(\rho))$, $\lambda(\rho)$ *being close to* 1/2, *such that* $\mathbf{x}^0 = (\rho\mu^0, -\rho)$ *is a fixed point of the Poincaré map defined above.*

(ii) $(\lambda(\rho), \mu^0(\rho))$ *depends smoothly on* ρ. *The value* $\lambda_0 = 1/2$ *of the parameter is a bifurcation value of a branch of non-trivial periodic solutions of system* (0.1).

(iii) *For each pair* $(\lambda, \mathbf{x})$ *such that* $\mathbf{x}$ *is a periodic solution of* (0.1) *for the value* λ *of the parameter,* $(\mathbf{x}, \lambda)$ *being close to* $(1/2, 0)$, *there exists a unique point* (ρ, μ^0) *with* $\lambda = \lambda(\rho)$, $\mu^0 = \mu^0(\rho)$, *and a point* $\bar{x}$ *on the orbit of* $x(t)$ *such that* $\bar{x} = (\rho\mu^0, -\rho, 0)$.

THEOREM 2.2. *Consider the system*

$$\frac{\mathbf{dx}}{\mathbf{dt}} = -\boldsymbol{\lambda}\mathbf{x} - \mathbf{F}(\mathbf{x}), \tag{0.1}$$

where λ *is a real parameter* $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)$,

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \mathbf{f}(x_2) \\ \mathbf{f}(x_3) \\ \mathbf{f}(x_1) \end{pmatrix},$$

where $\mathbf{f}$ *is an odd* $\mathbf{C}^1$ *function such that* $\mathbf{f}'(0)=1$, $x\mathbf{f}(x)>0$, *for* $x\neq 0$, *and* $(*)$ *holds. Let* $\mathbf{x}^0=(x_1^0, x_2^0, 0)$ *be an initial value for system* (0.1) *such that* $x_1^0>0>x_2^0$. *Set* $x_1^0=\rho\mu^0$ *and* $x_2^0=-\rho$.

Then, there exists $\rho_0>0$ *such that for every* ρ, $0<\rho<\rho_0$, *there exists* $\lambda(\rho)$ *so that the Poincaré operator associated to Eq.* (0.1), *with* $\lambda=\lambda(\rho)$, *has at least one non-trivial fixed point. This fixed point corresponds to a periodic solution of* (0.1) *with* $\lambda=\lambda(\rho)$.

These theorems follow directly from the study of the Poincaré operator that we will construct in Sections 1 and 2 below. The solutions we obtain have the following property: there exists a geometric transformation denoted by κ (a permutation of the components followed by the central symmetry), such that, if T is the period of the solution x, then we have $\mathbf{x}(t+T/6)=\kappa\mathbf{x}(t)$.

The local Hopf bifurcation theorem applies in the situation described in Theorem 2.1 and it could be used to prove that the solutions have the above-said property. But in fact the proof is comparable to the one we are presenting here and moreover it does not work in the C^1 framework (see the end of Section 2).

In Section 3 we prove the following global bifurcation result using the Browder ejective fixed point theorem.

THEOREM 3.1. *Consider the equation*

$$\frac{d\mathbf{x}}{dt}=-\lambda\mathbf{x}-\mathbf{F}(\mathbf{x}), \tag{0.1}$$

where

$$\mathbf{F}(\mathbf{x})=\begin{pmatrix}\mathbf{f}(x_2)\\ \mathbf{f}(x_3)\\ \mathbf{f}(x_1)\end{pmatrix}$$

for $\mathbf{x}=(x_1, x_2, x_3)$. *We assume that* $\mathbf{f}$ *is odd, monotone non-decreasing in* $\mathbb{R}^+$, *and differentiable with* $\mathbf{f}'(0)=1$. *We assume moreover that* $x\mathbf{f}(x)>0$ *for* $x\neq 0$, *and that for every* $a>0$ *there exists* $\mathbf{R}>0$ *such that* $\mathbf{f}(\mathbf{R})\leqslant \mathbf{aR}$.

Then for every $\lambda\in\,]0, 1/2]$ *there exists at least one non-trivial fixed point of the Poincaré operator defined in Section* 1. *These fixed points are the initial values of periodic solutions of Eq.* (0.1).

Let us end this introduction by a summary of the hypotheses we will use:

> We assume that $\mathbf{f}$ is an odd $\mathbf{C}^1$ function such that $\mathbf{xf(x)}>0$ for $\mathbf{x}\neq 0$. Moreover we suppose that $\sigma(\mathbf{x})\geqslant 0$, $\mathbf{x}\neq 0 \Rightarrow g(x)>0$. (H)

Occasionally, we will assume more on f, that it is C^2 or non-decreasing, or less, especially in the next section where most of the results do not require f to be odd.

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We will first give two results which indicate that we are not restricting the generality by looking for periodic solutions which cross coordinate planes, because indeed they all do that.

LEMMA 1.1. *Assume that* $xf(x)>0$, $x\neq 0$, $\lambda>0$. *Let* $\mathbf{x}$ *be a non-trivial periodic solution of* (1). *Choose any pair of components, say* (x_1, x_2). *Then either* $x_1(t)\,x_2(t)<0$, *for all* t, *or there exists* t_0 *such that* $(x_1(t_0)\,x_2(t_0)=0$.

Proof. We proceed by contradiction. Assuming that neither one of the properties holds, we conclude that $x_1(t)\,x_2(t)>0$ for all t. This implies that $\dot{x}_1(t)<0$ for all t, since $\dot{x}_1(t)=-[\lambda x_1(t)+f(x_2(t))]$, and λx_1 has the same sign as $f(x_2)$. x_1 being a C^1-periodic function its derivative takes the value zero at some point. ∎

LEMMA 1.2. *We assume that the condition* (∗) *on* g *holds. Let* $\mathbf{x}$ *be a non-trivial periodic solution of* (1). *Then* $\sigma(\mathbf{x})(t)<0$ *for all* t, *and there exists* t_0 *such that one cmponent of* $\mathbf{x}$, *say* x_i, *vanishes at* t_0.

For any such t_0 *we have* $(dx_i/dt)(t_0)\neq 0$, *the product of the other two components being* <0.

Proof. Let $\mathbf{x}$ be a non-trivial periodic solution and consider a point $\bar{t}$ where the periodic function $\sigma(t)$ achieves its maximum, so that $(d/dt)\,\sigma(\mathbf{x})(\bar{t})=0$. Note that

$$\begin{aligned}\frac{d}{dt}(\sigma(\mathbf{x}))(t) &= -2\lambda\sigma(\mathbf{x})-\sum_{i\neq j}\{x_j f(x_{[i+1]})+x_i f(x_{[j+1]})\}\\ &= -2\lambda\sigma(\mathbf{x})-g(\mathbf{x}).\end{aligned}$$

Since $\mathbf{x}$ is non-trivial, we have $\bar{\mathbf{x}}(t)\neq 0$. Assume that $\sigma(\mathbf{x})(\bar{t})\geqslant 0$, then $g(\mathbf{x})(\bar{t})>0$, since $\mathbf{x}(\bar{t})\neq 0$ and the hypothesis (∗) holds. Thus, $(d/dt)\,\sigma(\mathbf{x})(\bar{t})<0$, a contradiction. So $\sigma(\mathbf{x})(\bar{t})<0$ which means that $\sigma(\mathbf{x}(t)<0$, for all t. We will now see that one of the components of $\mathbf{x}$ takes the value zero at some point t_0. We proceed by contradiction.

Assuming that none of the components takes the value zero, we conclude that each component keeps a constant sign. Hence there are at least two of them having the same sign, their product being positive for all t. This contradicts Lemma 1.1. ∎

Let us finally prove the third part of the lemma: if at t_0 one of the components of $\mathbf{x}$, say x_1, vanishes, then $\sigma(\mathbf{x})(t_0)=x_2(t_0)\,x_3(t_0)<0$, simply because $\sigma(x)<0$. Moreover, $\dot{x}_1(t_0)=-f(x_2(t_0))$ is $\neq 0$ since $x_2(t_0)\neq 0$. This completes the proof. ▮

In view of Lemma 1.2 we are justified in looking at solutions generated by initial data belonging to one of the quadrants of a coordinate plane, with components of opposite signs. For example, we will assume that

$$x_1(0)>0>x_2(0) \qquad \text{and} \qquad x_3(0)=0.$$

Remark 1.1. Let us emphasize that the hypothesis (*) on g implies that for any solution x of (0.1), if $\sigma(x(t_0))<0$, for some t_0, then $\sigma(\mathbf{x})(t)<0$, for all $t\geqslant t_0$. In fact if $\sigma(\mathbf{x})$ changes sign it must vanish for some value say t_1 of t. This implies that $\sigma'(\mathbf{x}(t_1))=-2\lambda\sigma(\mathbf{x}(t_1))-g(\mathbf{x})<0$ and this means that $\sigma(\mathbf{x}))<0$ for t close enough to t_1. Hence $\sigma(\mathbf{x})$ keeps the same sign around t_1. More generally if for some t_0 we have $x_1(t_0)=x_3(t_0)=0$, $x_2(t_0)<0$, then $\sigma(\mathbf{x}(t))<0$ for $t>t_0$. In order to prove this claim, we may assume that $t_0=0$. Then

$$x_1(t)=-\int_0^t e^{-\lambda(t-s)}f(x_2(s))\,ds>0,$$

and

$$x_3(t)=-\int_0^t e^{-\lambda(t-s)}f(x_2(s))\,ds<0$$

hence for t_2 small enough, we have $\sigma(\mathbf{x}(t_2))<0$ and by the first part of the proof, we have $\sigma(\mathbf{x}(t))<0$ for $t>t_2$. So the solutions we will be considering from now on verify the property $\sigma(x(t))<0$, for $t>0$.

LEMMA 1.3. *Assume the same hypotheses as in Lemma* 1.2. *Let* $\mathbf{x}(t)=(x_1(t), x_2(t), x_3(t))$ *be an arbitrary solution of system* (0.1). *Assume that for each* i, $1\leqslant i\leqslant 3$, $x_i(t)$ *keeps a constant sign for* $t\geqslant 0$.

Then $\mathbf{x}(t)$ *tends to zero as* t *tends to infinity.*

Proof. We have two different situations: (1) all components have the same sign; (2) to of them are of one sign, the other one has the other sign. In both cases, there is at least one index, say j, such that x_j and $x_{[j+1]}$ have the sma sign. Thus, keeping in mind that $\lambda>0$, dx_j/dt has a constant sign, and in fact $x_j(t)\cdot(dx_j/dt)<0$.

As an example, suppose $x_j(t)>0$. In that case, $x_j(t)$ decreases and tends asymptotically to a limit x_j^*, $x_j^*\geqslant 0$. Another consequence we can draw from the equation verified by x_j is that $\lambda x_j+f(x_{[j+1]})\in L^1(0,+\infty)$. Because x_j and $f(x_{[j+1]})$ have the same sign, and $\lambda>0$, we conclude that $x_j\in L^1(0,+\infty)$, so $x_j^*=0$.

From $x_j(t) \to 0$ and the fact that f is differentiable at zero, we deduce that $f(x_j) \in L^1(0, +\infty)$. Thus, the equation verified by $x_{[j-1]}$ reads as $(d/dt)\, x_{[j-1]}(t) = -\lambda x_{[j-1]}(t) + h(t)$, with $h \in L^1(0, +\infty)$.

A straightforward application of the variation of constants formula leads to the conclusion that $x_{[j-1]}(t)$ tends to a finite limit at $+\infty$. Denote $x^*_{[j-1]}$ as this limit.

On the other hand, we can write $x_{[j-1]}$ as

$$x_{[j-1]}(t) = e^{-\lambda t}\bar{x} + \int_0^t e^{-\lambda(t-s)} h(s)\, ds,$$

where $\bar{x} = x_{[j-1]}(0)$. The right hand side is a sum of two L^1 functions (the second one is the convolution of two L^1 functions). So $x_{[j-1]} \in L^1(0, +\infty)$ and therefore $x_{[j-1]} = 0$. The same argument can be repeated for $x_{[j-2]}$, and in fact would apply in a more general situation. The proof of Lemma 1.3 is complete. ▮

Remark 1.2. A consequence of Lemma 1.3 is that at least one of the components of a periodic solution takes the value zero at some point. This has already been proved in Lemma 1.2 in a different way.

LEMMA 1.4. *Assume the same hypotheses as in Lemma* 1.2, *and* $\lambda > 0$, $\lambda \neq 1/2$. *Let* $\mathbf{x}(t)$ *be a solution of system* (0.1) *with initial value* $(x_1^0, x_2^0, 0)$ *such that* $x_1^0 \geq 0 > x_2^0$. *Then there exists a positive time such that one of the components of* $\mathbf{x}$ *is equal to zero. The first positive time* τ *for which this holds is such that* $x_2(\tau) = 0$, $x_3(\tau) < 0$, $x_1(\tau) > 0$.

Proof. If all components of $\mathbf{x}(t)$ are different from zero for all $t > 0$ then by Lemma 1.3, $\mathbf{x}(t)$ tends to 0 as t tends to $+\infty$. Let us look for a moment at the linearized system of (0.1) around the zero solution.

The corresponding matrix is

$$\begin{pmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & -1 \\ -1 & 0 & -\lambda \end{pmatrix} = -\lambda I - \mathbf{A}.$$

We observe that

$$\frac{d}{dt}(x_1 + x_2 + x_3)(t) = -(\lambda + 1)(x_1 + x_2 + x_3)(t),$$

hence

$$(x_1 + x_2 + x_3)(t) = e^{-(\lambda+1)t}(x_1^0 + x_2^0 + x_3^0).$$

This means that the subspace $Y=\{(x_1, x_2, x_3): x_1=x_2=x_3\}$ is invariant by the equation $\dot{y}=-(\lambda+1)y$.

The other two eigenvalues of the matrix $-\lambda I-\mathbf{A}$ are $\mu=-\lambda-j$ and $\mu=-\lambda-j^2$, where j is one of the complex cubic roots of 1, $1+j+j^2=0$. Re $\mu \neq 0$ if and only if $\lambda \neq 1/2$ and then we are in a hyperbolic situation. Let us first look at $\lambda<1/2$ (or Re $\mu>0$). We now come back to the study of $\mathbf{x}(t)$. $\mathbf{x}(t)$ belongs to the stable manifold of (0.1) at zero, since $\mathbf{x}(t)\to 0$ as $t\to+\infty$. This stable manifold is tangent at zero to the eigenspace associated to the negative eigenvalue. Hence there exists a nonzero constant C such that

$$\mathbf{x}(t)=Ce^{-(\lambda+1)t}\begin{pmatrix}1\\1\\1\end{pmatrix}+o(e^{-(\lambda+1)t}), \qquad \text{as} \quad t\to+\infty.$$

Hence the components of $\mathbf{x}(t)$ are of the same sign for t large enough. But we have $x_1^0x_2^0\leqslant 0$ and $x_3^0=0$, which implies that $\sigma(\mathbf{x})(0)\leqslant 0$, and by Remark 1.1, $\sigma(\mathbf{x})(t)<0$, for every $t\geqslant 0$.

On the other hand, assuming that all the components of $\mathbf{x}$ have the same sign at a time t would imply $\sigma(\mathbf{x})(t)>0$. This leads to a contradiction.

Let us now turn to the situation where $\lambda>1/2$. The origin is asymptotically stable in that case. The dominant eigenvalues are complex. Most of the solutions have a principal part which is a combination of exponential solutions with complex exponent. These solutions change signs. The only solutions which do not change signs are tangent to the eigenspace generated by the vector (1, 1, 1), thus all of their components are of the same sign for t large enough. The conclusion of this part is that at least one of the components of $\mathbf{x}$ takes the value zero at some positive time.

Let us now prove the second part of the lemma: that is, we want to prove that x_2 is the first component to vanish.

Note first that we have $(d/dt)\,x_3(0)<0$, which means that for any $\alpha>0$, there exists α', $0<\alpha'<\alpha$, such that $x_1(\alpha')\,x_2(\alpha')\,x_3(\alpha')\neq 0$. Therefore, the infimum of positive times t for which one of the components cancels is positive. Denote it by τ.

Assuming that $x_1(\tau)=0$ would imply that $\sigma(\mathbf{x})(\tau)=x_2(\tau)\,x_3(\tau)>0$, which cannot be, since $\sigma(\mathbf{x})(0)<0$, so according to Remark 1.1, $\sigma(x)(t)<0$, for $t\geqslant 0$.

On the other hand, assuming $x_3(\tau)=0$, would imply that

$$x_3(t)=\int_t^\tau e^{-\lambda(t-s)}f(x_1(s))\,ds, \qquad \text{for} \quad 0\leqslant t<\tau,$$

so $x_3(t)>0$, for $0\leqslant t<\tau$, which yields another contradiction. Hence we must have $x_2(\tau)=0$. The proof of Lemma 1.4 is complete. ∎

In the sequel, we will use the following definition

DEFINITION 1.1. For any solution $\mathbf{x}(t)$ of system (1) starting from $\mathbf{x}^0 \in \mathbb{R}^+ \times \mathbb{R}^-$, we call the (*first*) return time, the time needed by $\mathbf{x}(t)$ to get back to $\mathbb{R}^+ \times \mathbb{R}^-$ for the first time.

Consequence. In the same way one can prove that the next component of $\mathbf{x}(t)$ to vanish is x_1, x_3 being negative and x_2 positive. More precisely one can build the following table which indicates the rule of the change of signs:

The Poincaré Map

x_1	x_2	x_3
+	−	0
+	0	−
0	+	−
−	0	+
0	−	+
+	−	0

Before we construct the Poincaré map, it may be convenient to discuss at this point a remarkable property of equations of type (0.1): invariance with respect to a set of transformations of the state space. More specifically, we introduce the following maps:

$$T: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix},$$

the circular permutation of the components, and $S: \mathbf{x} \to -\mathbf{x}$, the symmetry with respect to the origin. T generates a group with 3 elements: $\{\mathrm{Id}, T, T^2\}$, while S generates a group with 2 elements $\{\mathrm{Id}, S\}$. Combining S and T together by composition product, we obtain a group of 6 elements generated by

$$\kappa = ST^2 \qquad \text{or its inverse } \kappa^{-1} = ST.$$

It is easy to check that (0.1) is invariant by T. This means that if we denote by $\mathbf{x}(t, \mathbf{x}^0)$ the solution operator, we have

$$T\mathbf{x}(t, \mathbf{x}^0) = \mathbf{x}(t, T\mathbf{x}^0).$$

If we add the condition that f is odd, then system (0.1) will be invariant by S too, therefore it will be invariant by κ and by any of its iterates. We will come back to that in Section 2. We are now ready to derive the Poincaré map for the linear system (0.5). Given an arbitrary point $\mathbf{x}^0$ on the plane $x_3=0$, that is, $\mathbf{x}^0=(x_1^0, x_2^0, 0)$, such that $x_1^0>0>x_2^0$, we know that the solution starting from $\mathbf{x}^0$ at $t=0$ hits the plane $x_2=0$ some time τ later. This builds up a first mapping

$$Q: \mathbb{R}^+ \times \mathbb{R}^- \times \{0\} \to \mathbb{R}^+ \times \{0\} \times \mathbb{R}^-,$$
$$Q(x_1^0, x_2^0, 0) = (x_1(\tau), 0, x_3(\tau)).$$

Then by composition with κ^{-1}, we have a map $\kappa^{-1} \circ Q: \mathbb{R}^+ \times \mathbb{R}^- \times \{0\} \to \{0\} \times \mathbb{R}^+ \times \mathbb{R}^-$. The Poincaré map is just the restriction of $\kappa^{-1} \circ Q$ to the first two components

$$\Pi: \mathbb{R}^+ + \mathbb{R}^-,$$
$$\Pi(x_1^0, x_2^0) = -(x_3(\tau), x_1(\tau)).$$

Now we will solve system (0.5) with initial value $\mathbf{x}^0=(x_1^0, x_2^0, 0)$ with $x_1^0>0>x_2^0$ and prove that fixed points of Π do exist.

We observe that the matrix $\mathbf{A}$ is such that $\mathbf{A}^3=\mathrm{Id}$, so we can write

$$e^{tA} = \sum_{k=0}^{\infty} \left(\frac{A^{3k}t^{3k}}{(3k)!} + \frac{A^{3k+1}t^{3k+1}}{(3k+1)!} + \frac{A^{3k+2}t^{3k+2}}{(3k+2)!} \right)$$
$$= \left[\sum_{k=0}^{\infty} \frac{t^{3k}}{(3k)!} \right) I + \left(\sum_{k=0}^{\infty} \frac{t^{3k+1}}{(3k+1)!} \right) \mathbf{A} + \left(\sum_{k=0}^{\infty} \frac{t^{3k+2}}{(3k+2)!} \right) \mathbf{A}^2$$
$$= S_0(-t)I + S_1(-t)\mathbf{A} + S_2(-t)\mathbf{A}^2 = \begin{pmatrix} S_0(-t) & S_1(-t) & S_2(-t) \\ S_2(-t) & S_0(-t) & S_1(-t) \\ S_1(-t) & S_2(-t) & S_0(-t) \end{pmatrix},$$

where $S_0(-t)=\sum_{k=0}^{\infty} t^{3k}/(3k)!$; $S_1(-t)=\sum_{k=0}^{\infty} t^{3k+1}/(3k+1)!$; $S_2(-t)=\sum_{k=0}^{\infty} t^{3k+2}/(3k+2)!$. Hence the solution of system (0.5) with initial value $\mathbf{x}^0$ is given by

$$\mathbf{x}(t) = e^{-\lambda t} \begin{pmatrix} S_0(t) & S_1(t) & S_2(t) \\ S_2(t) & S_0(t) & S_2(t) \\ S_1(t) & S_2(t) & S_0(t) \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix}.$$

One can check easily that S_i verifies the following third order differential equation $S_i''' + S_i = 0$, for $i=1, 2, 3$. Solving these equations we get

$$S_0(t) = \frac{1}{3} e^{-t} + \frac{2}{3} e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right),$$

$$S_1(t)=\frac{1}{3}e^{-t}-\frac{1}{3}e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right),$$

$$S_2(t)=\frac{1}{3}e^{-t}-\frac{1}{3}e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{1}{\sqrt{3}}e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

We note that $S_1'=-S_0$, $S_2'=-S_1$, and $S_0'=-S_2$.

The existence of fixed points of the Poincaré operator Π for the linear equation (0.5) is established in the following proposition.

PROPOSITION 1.1. *Consider system* (0.5) *with initial data* $\mathbf{x}^0=(x_1^0, x_2^0, 0)$ *with* $x_1^0>0>x_2^0$. *Then the Poincaré map* $\Pi(x_1^0, x_2^0)=-(x_3(\tau), x_1(\tau))$ *has nonzero fixed points for* $\lambda=1/2$ *and* $\tau^*=2\pi/3\sqrt{3}$.

Proof. The minimal time τ, $\tau>0$ such that $x_2(\tau)=0$, satisfies the equations

$$x_1(\tau)=e^{-\lambda\tau}(S_0(\tau)x_1^0+S_1(\tau)x_2^0),$$
$$0=e^{-\lambda\tau}(S_2(\tau)x_1^0+S_0(\tau)x_2^0),$$
$$x_3(\tau)=e^{-\lambda\tau}(S(\tau)x_1^0+S_2(\tau)x_2^0).$$

Using the second equation we find that

$$x_1(\tau)=e^{-\lambda\tau}\frac{S_1(\tau)S_2(\tau)-(S_0(\tau))^2}{S_2(\tau)}x_2^0,$$

$$x_3(\tau)=e^{-\lambda\tau}\frac{S_0(\tau)S_1(\tau)-(S_2(\tau))^2}{S_0(\tau)}x_1^0.$$

Hence a fixed point (x_1^0, x_2^0) of Π corresponds to the equations

$$e^{-\lambda\tau}\frac{S_1(\tau)S_2(\tau)-(S_0(\tau))^2}{S_2(\tau)}=1,$$

$$e^{-\lambda\tau}\frac{S_0(\tau)S_1(\tau)-(S_2(\tau))^2}{S_0(\tau)}=1.$$

In particular, this leads to

$$(S_2(\tau))^3-S_0(\tau)S_1(\tau)S_2(\tau)=(S_0(\tau))^3-S_0(\tau)S_1(\tau)S_2(\tau).$$

This implies

$$S_0(\tau)=S_2(\tau) \quad\text{and}\quad e^{-\lambda\tau}(S_2(\tau)-S_1(\tau))=1. \tag{1.1}$$

The first relation in (1.1) means that $\cos((\sqrt{3}/2)\tau) = (1/\sqrt{3})\sin((\sqrt{3}/2)\tau)$. The first corresponding value for $\tau > 0$ is $\tau = 2\pi/3\sqrt{3}$, the other relation in (1.1) gives $\lambda = 1/2$. ▮

Remark 1.3. What does it mean to find a fixed point of Π? We will conclude this section by answering this question. Assume here that f is odd. We are looking at the nonlinear situation. Suppose that for some $\mathbf{x}^0 \neq 0$, we have

$$\Pi \mathbf{x}^0 = \mathbf{x}^0.$$

In terms of κ and Q, this means that $(\kappa^{-1} \circ Q)(\mathbf{x}^0) = \mathbf{x}^0$.

$Q\mathbf{x}^0 = \mathbf{x}^\tau$, where τ is the first time that the solution reaches the plane $x_3 = 0$. In the introduction of this section, we saw that for all $t \geqslant 0$, $\kappa^{-1}\mathbf{x}(t) = \mathbf{x}(t, \kappa^{-1}\mathbf{x}^0)$. Finally we get $\mathbf{x}(\tau, \kappa^{-1}\mathbf{x}^0) = \mathbf{x}^0$. This implies, by the uniqueness property of the solutions and the equation being time-independent, that

$$\mathbf{x}(t+\tau, \kappa^{-1}\mathbf{x}^0) = \mathbf{x}(t, \mathbf{x}^0).$$

Applying κ^{-1} once more, we obtain $\mathbf{x}(t+\tau, \kappa^{-2}\mathbf{x}^0) = \mathbf{x}(t, \mathbf{x}^0)$, thus $\mathbf{x}(2\tau, \kappa^{-2}\mathbf{x}^0) = \mathbf{x}(\tau, \kappa^{-1}\mathbf{x}^0) = \mathbf{x}^0$. More generally, we have $\mathbf{x}(j\tau, \kappa^{-j}\mathbf{x}^0) = \mathbf{x}^0$, thus $\mathbf{x}(6\tau, \mathbf{x}^0) = \mathbf{x}^0$, since $\kappa^{-6} = \mathrm{Id}$.

So solving the fixed point problem related to the Poincaré operator associated to (0.1) amounts to finding periodic solutions of (0.1), periodic solutions invariant by the same transformations that leave the equation invariant.

2

In this section we exploit the results of Section 1 in the study of periodic solutions of the nonlinear system

$$\frac{d\mathbf{x}}{dt} = (\lambda I - \mathbf{A})\mathbf{x} - \mathbf{G}(\mathbf{x}), \tag{2.1}$$

$\mathbf{x} = (x_1, x_2, x_3)$. As indicated earlier $\mathbf{A}$ is the linearization around zero, of the function $\mathbf{F}$ given in (0.1). First we assume that besides the general hypothesis (**H**), f is C^2, which implies that $\mathbf{G}$ is C^2. From the definition of $\mathbf{G}$, we have $\mathbf{G}(\mathbf{x}) = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to 0$.

We look for fixed points of the corresponding Poincaré operator. Here the idea is to determine eigenvectors of the Poincaré map and among them those having 1 as an eigenvalue. The solution of (2.1) with initial value $\mathbf{x}^0$ verifies

$$\mathbf{x}(t) = e^{(-\lambda - A)t}\mathbf{x}^0 - \int_0^t e^{(-\lambda - A)(t-s)}\mathbf{G}(\mathbf{x}(s))\,ds.$$

We set

$$\omega(\lambda, \mathbf{x}^0, t) = -\int_0^t e^{(-\lambda - A)(t-s)}\mathbf{G}(\mathbf{x}(s))\,ds.$$

The function ω is C^2 and $D_{\mathbf{x}^0}\omega(\lambda, 0, t) = 0$. As an initial value, we take $\mathbf{x}^0$ such that $x_1^0 > 0 > x_2^0$, $x_3^0 = 0$.

The equation $x_2(\tau) = 0$ which gives the "return time" reads

$$\begin{aligned} x_2(\tau) &= e^{-\lambda\tau}(e^{-A\tau}\mathbf{x}^0)_2 + \omega_2(\lambda, \mathbf{x}^0, \tau), \\ &= e^{-\lambda\tau}(S_2(\tau)x_1^0 + S_0(\tau)x_2^0) + \omega_2(\lambda, \mathbf{x}^0, \tau) = 0. \end{aligned} \tag{2.2}$$

Let us set $x_1^0 = \rho\mu^0$, $x_2^0 = -\rho$. We now define $\Omega(\lambda, \rho, \mu^0, t)$ by

$$\Omega(\lambda, \rho, \mu^0, t) = \frac{1}{\rho}\,\omega(\lambda, \rho\mu^0, -\rho, 0, t).$$

Note that Ω is C^1 and that $\Omega(\lambda, \rho, \mu^0, t) = o(1)$ as ρ tends to zero since $\omega = o(\rho)$, locally uniformly with respect to the other parameters. In view of these notations, the equation $x_2(\tau) = 0$ is equivalent to

$$e^{-\lambda\tau}[\mu^0 S_2(\tau) - S_0(\tau)] + \Omega_2(\lambda, \rho, \mu^0, \tau) = 0. \tag{2.3}$$

We have the following lemma which describes τ as a function of λ and μ^0 near τ^* (defined in Proposition 1.1).

LEMMA 2.1. *The equation* $x_2(\tau) = 0$ *defines locally* τ *as a* C^1 *function* $\tau(\lambda, \rho, \mu^0)$ *such that* $\tau(\lambda, 0, 1) = \tau^*$.

Proof. Differentiating formula (2.3) with respect to τ at $(\lambda, 0, 1, \tau^*)$ we get

$$e^{-\lambda\tau^*}(S_2'(\tau^*) - S_0'(\tau^*)) + \frac{\partial}{\partial\tau}\Omega_2(\lambda, 0, 1, \tau^*) = e^{-\lambda\tau^*}(S_2'(\tau^*) - S_0'(\tau^*)) \neq 0.$$

By the implicit function theorem, Eq. (2.2) can be solved for τ. This defines a function $\tau(\lambda, \rho, \mu^0)$ with $\tau(\lambda, 0, 1) = \tau^*$. Moreover τ is of class C^1. ∎

Remark 2.1. For $\rho = 0$, Eq. (2.3) becomes $\mu^0 S_2(\tau) - S_0(\tau) = 0$, since Ω_2 is $o(1)$ as $\rho \to 0$. Thus $\tau(\lambda, 0, \mu^0)$ is a function of μ^0 only, which we still denote by $\tau(\mu^0)$. In particular $\tau(\lambda, 0, 1) = \tau(1) = \tau^*$.

Let us introduce the following ratio $\mu^\tau = -x_3^\tau/x_1^\tau = \mu(\tau, \lambda, \rho, \mu^0)$ where τ is the return time for the solution. Then we have the following lemma

LEMMA 2.2. *The relation $\mu^\tau = \mu^0$, with $\tau = \tau(\lambda, \rho, \mu^0)$, as given by Lemma 2.1, determines μ^0 as a smooth function of λ and ρ near $\lambda = 1/2$ and $\rho = 0$, such that $\mu^0(1/2, 0) = 1$.*

Proof. We use once again the implicit function theorem. For this purpose, we will prove that $(\partial\mu^\tau/\partial\mu^0)(1/2, 0, 1) \neq 1$.

$$\mu^\tau = -\frac{e^{-\lambda\tau}(e^{-A\tau}\mathbf{x}^0)_3 + \omega_3(\lambda, \mathbf{x}^0, \tau)}{e^{-\lambda\tau}(e^{-A\tau}\mathbf{x}^0)_1 + \omega_1(\lambda, \mathbf{x}^0, \tau)},$$

$$= -\frac{e^{-\lambda\tau}(S_1(\tau)\,\mu^0 - S_2(\tau)) + \Omega_3(\lambda, \rho, \mu^0, \tau)}{e^{-\lambda\tau}(S_0(\tau)\mu^0 - S_1(\tau)) + \Omega_1(\lambda, \rho, \mu^0, \tau)}.$$

Differentiating with respect to μ^0, we can let $\rho = 0$ since μ^τ is C^1 in a neighborhood of $(1/2, 0, 1)$. Now, for $\rho = 0$, μ^τ is reduced to

$$\mu^\tau(\lambda, 0, \mu^0) = -\frac{S_1(\tau)\mu^0 - S_2(\tau)}{S_0(\tau)\mu^0 - S_1(\tau)}.$$

We have to differentiate this function with respect to μ^0.

For the convenience of computations we set $\Phi(\tau, \mu^0) = \mu^\tau(\lambda, 0, \mu^0)$ where we remind the reader that τ is $\tau(\lambda, \rho, \mu^0)$. We have

$$\frac{d}{d\mu^0}[\Phi(\tau(\mu^0), \mu^0)] = \frac{\partial\Phi}{\partial\tau}(\tau(\mu^0), \mu^0)\cdot\tau'(\mu^0) + \frac{\partial\Phi}{\partial\mu^0}(\tau(\mu^0), \mu^0).$$

Let us calculate each term.

$$\frac{\partial\Phi}{\partial\tau}(\tau, \mu^0) = -\frac{\begin{bmatrix}(S_1'(\tau)\mu^0 - S_2'(\tau))(S_0(\tau)\mu^0 - S_1(\tau)) \\ -(S_0'(\tau)\mu^0 - S_1'(\tau))(S_1(\tau)\mu^0 - S_2(\tau))\end{bmatrix}}{(S_0(\tau)\mu^0 - S_1(\tau))^2}$$

In Section 1, we saw that $S_1' = -S_0$, $S_2' = -S_1$, and $S_0' = -S_2$.

We also know that $S_0(\tau^*) = S_2(\tau^*)$ (see formula (1.1)). Hence we have

$$\frac{\partial\Phi}{\partial\tau}(\tau, \mu^0) = -\frac{\begin{bmatrix}-S_0(\tau)\,\mu^0 + S_1(\tau))(S_0(\tau)\,\mu^0 - S_1(\tau)) \\ -(-S_2(\tau)\mu^0 + S_0(\tau))(S_1(\tau)\mu^0 - S_2(\tau))\end{bmatrix}}{(S_0(\tau)\mu^0 - S_1(\tau))^2}.$$

So

$$\frac{\partial\Phi}{\partial\tau}(\tau^*, 1) = 1,$$

since $S_0(\tau^*) - S_2(\tau^*)\mu^0 = 0$ for $\mu^0 = 1$.

$$\frac{\partial \Phi}{\partial \mu^0}(\tau, \mu^0) = -\frac{S_1(\tau)[-S_1(\tau) + S_0(\tau)\mu^0] - S_0(\tau)[S_1(\tau)\mu^0 - S_1(\tau)]}{(S_0(\tau)\mu^0 - S_1(\tau))^2}$$

$$= \frac{S_1^2 - S_0 S_2}{(S_0\mu^0 - S_1)^2}, \qquad \text{where } S_i \text{ stands for } S_i(\tau).$$

Hence,

$$\frac{\partial \Phi}{\partial \mu^0}(\tau^*, 1) = -\frac{S_0(\tau^*) + S_1(\tau^*)}{S_0(\tau^*) - S_1(\tau^*)}.$$

Let us calculate $\tau'(\mu^0)$ using the equation $\mu^0 S_2 - S_0 = 0$. We have $S_2 + (\mu^0 S_2' - S_0')\cdot\tau'(\mu^0) = 0$, or $S_2 + (S_2 - S_1\mu^0)\,\tau'(\mu^0) = 0$, since $S_0' = -S_2$ and $S_2' = -S_1$. For $\mu^0 = 1$ and $\tau = \tau^*$, we get

$$\tau'(1) = \frac{S_2(\tau^*)}{S_1(\tau^*) - S_2(\tau^*)} = \frac{S_0(\tau^*)}{S_1(\tau^*) - S_0(\tau^*)}.$$

Finally

$$\frac{d\Phi}{d\mu^0}(\tau^*, 1) = \tau'(1) + \frac{\partial \Phi}{\partial \mu^0}(\tau^*, 1)$$

$$= \frac{S_0(\tau^*)}{S_1(\tau^*) - S_0(\tau^*)} - \frac{S_0(\tau^*) + S_1(\tau^*)}{S_0(\tau^*) - S_1(\tau^*)}$$

$$= \frac{2S_0(\tau^*) + S_1(\tau^*)}{S_1(\tau^*) - S_0(\tau^*)} = e^{-\tau^*}.$$

So, $(\partial\Phi/\partial\mu^0)(\tau^*, 1) \neq 1$. So $(\partial\mu^\tau/\partial\mu^0)(1/2, 0, 1) \neq 1$.

We can apply the implicit function theorem to solve the equation $\mu^\tau = \mu^0$. This proves Lemma 2.2. ∎

The function $\mu^0(\lambda, \rho)$ gives the direction of the initial vector $x^0 = (\mu^0\rho, -\rho)$ which is nonlinear eigenvector of the Poincaré operator, the corresponding eigenvalue is $\alpha(\lambda, \rho) = -x_3^\tau / x_1^0$.

Fixed points of the Poincaré operator will be obtained by solving the equation $\alpha(\lambda, \rho) = 1$.

LEMMA 2.3. *The equation $\alpha(\lambda, \rho) = 1$ determines locally λ as a smooth function $\lambda(\rho)$ such that $\lambda(0) = 1/2$.*

Proof. Replacing the components x_1^0 and x_3^τ by their respective values and using $x_1^0 = \rho\mu^0$, $x_2^0 = -\rho$ we get

$$\alpha(\lambda, \rho) = -\frac{e^{-\lambda\tau}(S_1(\tau)x_1^0 + S_2(\tau)x_2^0) + \omega_3(\lambda, \mathbf{x}^0, \tau)}{x_1^0}$$
$$= -\frac{e^{\lambda\tau}(S_1(\tau)\mu^0 - S_2(\tau)) + \Omega_3(\lambda, \rho, \mu^0, \tau)}{\mu^0}$$

We will apply the implicit function theorem.

Since $\alpha(1/2, 0) = 1$, we have to calculate $(\partial\alpha/\partial\lambda)(1/2, 0)$. We can put $\rho = 0$ and differentiate $\alpha(\lambda, 0)$ with respect to λ, since $\alpha(\lambda, \rho)$ is C^1. We have

$$\alpha(\lambda, 0) = -\frac{e^{-\lambda\tau}(S_1(\tau)\mu^0 - S_2(\tau))}{\mu^0}.$$

Now $\tau = \tau(\mu^0)$ for $\rho = 0$, hence

$$\alpha(\lambda, 0) = -\frac{e^{-\lambda\tau(\mu^0)}(S_1(\tau(\mu^0))\mu^0 - S_2(\tau(\mu^0)))}{s^0}.$$

On the other hand, for $\rho = 0$, we have $\mu^\tau(\lambda, 0, \mu^0) = \Phi(\tau, \mu^0)$ [see Lemma 2.2]. The fixed point equation for μ^0 is then reduced to

$$\Phi(\tau(\mu^0), \mu^0) - \mu^0 = 0.$$

The function on the left hand side is locally invertible near $\mu^0 = 1$. So, $\mu^0 = 1$ is its only solution in some neighborhood of 1, and it is independent of λ since the equation does not depend on λ. Thus,

$$\mu^0(\lambda, 0) = \mu^0(1/2, 0) = 1.$$

So, the formula for $\alpha(\lambda, 0)$ reduces to

$$\alpha(\lambda, 0) = -e^{-\lambda\tau^*}[S_1(\tau^*) - S_2(\tau^*)] = e^{-(\lambda - 1/2)\tau^*},$$

which yields

$$\frac{\partial\alpha}{\partial\lambda}(1/2, 0) = \tau^* e^{-\lambda\tau^*}[S_1(\tau^*) - S_2(\tau^*)] = -\tau^* = -\frac{2\pi}{3\sqrt{3}}.$$

Hence $(\partial\alpha/\partial\lambda)(1/2, 0) \neq 0$.

The implicit function theorem applies. The equation $\alpha(\lambda, \rho) = 1$ gives a curve $\lambda(\rho)$ such that $\lambda(0) = 1/2$. ∎

Proof of Theorem 2.1. The curve $\lambda(\rho)$ obtained in Lemma 2.3 yields the fixed points of the Poincaré operator Π. These fixed points are initial values of periodic solutions of system (2.1). As a result of using implicit function theorems, they are uniquely determined in terms of ρ. This proves the statements of Theorem 2.1. ∎

Proof of Theorem 2.2. The proof of Theorem 2.2 is similar to the one of Theorem 2.1. But now f is assumed to be a C^1 function, only. So the function $\omega(\lambda, \mathbf{x}^0, t) = -\int_0^t e^{(-\lambda - A)(t-s)}\mathbf{G}(\mathbf{x}(s))\, ds$, introduced at the beginning of Section 2 is C^1 with respect to $\mathbf{x}^0$. It is at least C^2 with respect to t and it is C^∞ with respect to λ. Hence Ω is C^1 far away from 0, but it is only continuous at $\rho = 0$. Because of this lack of regularity, we begin by solving the eigenvalue equation $\alpha(\lambda, \rho, \mu^0) = 1$ for λ. This is done in Lemma 2.4 below. Then, in Lemma 2.5, we solve the eigenvector equation $\mu^\tau = \mu^0$. Theorem 2.2 is a direct consequence of Lemmas 2.4 and 2.5. We observe that the set of periodic solutions obtained in Theorem 2.2 is not necessarily a smooth curve parametrized by ρ. Keeping the notations of Lemma 2.2, we get the following lemma

LEMMA 2.4. *The equation* $\alpha(\lambda, \rho, \mu^0) = 1$ *determines locally* λ *as a smooth function* $\lambda(\rho, \mu^0)$ *such that* $\lambda(0, 1) = 1/2$.

Proof. We will prove that $(\partial\alpha/\partial\lambda)(1/2, 0, 1) \neq 0$. Then we apply the implicit function theorem as we did for Lemma 2.3. We may set $\rho = 0$ and $\mu^0 = 1$ before we differentiate. From the formulae

$$\alpha(\lambda, \rho, \mu^0) = -\frac{x_3^\tau}{x_1^0} = -\frac{e^{-\lambda\tau}(S_1(\tau)\mu^0 - S_2(\tau)) + \Omega_3(\lambda, \rho, \mu^0, \tau)}{\mu^0},$$

where τ is $\tau(\lambda, \rho, \mu^0)$, we get $\alpha(\lambda, 0, 1) = e^{-\lambda\tau^*}(S_1(\tau^*) - S_2(\tau^*))$. Hence $(\partial\alpha/\partial\lambda)(1/2, 0, 1) = -\tau^* \neq 0$.

The conclusion of Lemma 2.4 follows. ∎

LEMMA 2.5. *Assume that* (H) *holds. Then there exist* $\rho_0 > 0$, μ_1 *and* μ_2 *with* $\mu_1 < 1 < \mu_2$ *such that for every* $\rho > 0$, $\rho < \rho_0$ *there exists* μ^0, $\mu_1 < \mu^0 < \mu_2$ *satisfying the relation* $\mu^\tau = \mu^0$.

Proof. The equation $\mu^\tau = \mu^0$ reads

$$-\frac{e^{-\lambda\tau}(S_1(\tau)\mu^0 - S_2(\tau)) + \Omega_3(\lambda, \rho, \mu^0, \tau)}{e^{-\lambda\tau}(S_0(\tau)\mu^0 - S_1(\tau)) + \Omega_1(\lambda, \rho, \mu^0, \tau)} = \mu^0$$

or,

$$\begin{aligned} &e^{-\lambda\tau}(S_1(\tau)\mu^0 - S_2(\tau)) + \Omega_3(\lambda, \rho, \mu^0, \tau) \\ &\qquad + \mu^0[e^{-\lambda\tau}(S_0(\tau)\mu^0 - S_1(\tau)) + \Omega_1(\lambda, \rho, \mu^0, \tau)] = 0. \end{aligned}$$

We can write this equation in the form $S_0(\tau) \cdot (\mu^0)^2 - S_2(\tau) + o(1) = 0$ $(\rho \to 0)$. We know that $\tau = \tau(\mu^0) + o(1)$ as $\rho \to 0$. Let us set

$$\Psi_0(\mu^0) = S_0(\tau(\mu^0)) \cdot (\mu^0)^2 - S_2(\tau(\mu^0))$$

and

$$\Psi(\rho, \mu^0) = \Psi_0(\mu^0) + o(1) \qquad (\rho \to 0).$$

Then the equation $\mu^\tau = \mu^0$ becomes $\Psi(\rho, \mu^0) = 0$.

It may be worth noting that the function $\Psi(\rho, \mu^0)$ is only continuous in its variables while $\Psi_0(\mu^0)$ is C^1. The implicit function theorem does not apply . We will use a continuity argument to prove the existence of zeros of Ψ. Let us first look at Ψ_0. The only solution of $\Psi_0(\mu^0) = 0$ is $\mu^0 = 1$. Let us show that $\Psi_0'(1) \neq 0$. Direct calculations show that

$$\Psi_0'(\mu^0) = 2S_0(\tau(\mu^0)) - S_2(\tau(\mu^0)) \cdot \tau'(\mu^0) + S_1(\tau(\mu^0)) \cdot \tau'(\mu^0).$$

Now $\tau(1) = \tau^*$ and we know that $\tau'(1) = S_0(\tau^*)/(S_1(\tau^*) - S_0(\tau^*))$ (see the proof of Lemma 2.2). Let us denote $S_i(\tau^*)$ by S_i and replace $\tau'(1)$ by its value. We get

$$\begin{aligned}\Psi_0'(1) &= 2S_0 - S_2 \cdot \tau'(1) + S_1 \cdot \tau'(1),\\ &= 2S_0 - S_2 \cdot \frac{S_0}{S_1 - S_0} + S_1 \cdot \frac{S_0}{S_1 - S_0} = 3S_0,\end{aligned}$$

that is, $\Psi_0'(1) = 3S_0(\tau^*) \neq 0$.

Hence there exist $\mu_1 < 1$ and $\mu_2 > 1$ such that $\Psi_0(\mu_1) \cdot \Psi_0(\mu_2) < 0$. On the other hand there exists $\rho_0 > 0$ such that

$$\rho < \rho_0 \Rightarrow |o(1)| < \min(|\Psi_0(\mu_1)|, |\Psi_0(\mu_2)|).$$

For these values of ρ, $\Psi(\rho, \mu_i)$ is of the same sign as $\Psi_0(\mu_i)$, $i = 1, 2$. Thus $\Psi(\rho, \mu_1) \cdot \Psi(\rho, \mu_2) < 0$. Hence for each $\rho < \rho_0$ there exists μ^0, $\mu_1 < \mu^0 < \mu_2$ such that $\Psi(\rho, \mu^0) = 0$. That is, $\mu^\tau = \mu^0$. Thus completes the proof of Lemma 2.5. ∎

End of the Proof of Theorem 2.2. Lemmas 2.4 and 2.5 mean that the Poincaré operator has non-trivial fixed points. Theorem 2.2 follows. ∎

We end this section by comparing our method with the local Hopf bifurcation theorem. This theorem applies to our problem when f is assumed to be C^2. Let us indicate how to show tat the solutions obtained by using the Hopf theorem have the same symmetry properties as the solutions obtained in Theorem 2.1. The uniqueness property guaranteed by the Hopf theorem in the C^2 case added to the fact that our equation is invariant by κ proves that for every periodic solution $\mathbf{x}$ belonging to the local Hopf branch, there exists τ, such that $\kappa\mathbf{x}(t) = \mathbf{x}(t + \tau)$. This τ can be chosen so that $\tau \leqslant T$ where T is the period of $\mathbf{x}$. On the other hand, every periodic solution crosses at least once one of the coordinate planes in a region where the product of component is <0. More precisely, every periodic solution belonging to the local branch, crosses each of the six regions $\{x_i=0: x_j x_k<0\}$, $\{i, j, k\}=\{1, 2, 3\}$. Let $\mathbf{x}$ be a periodic solution belonging to the local (Hopf) branch. One can assume that $x_1(0)>0>x_2(0)$, $x_3(0)=0$. Let τ be the first time, $0 \leqslant \tau \leqslant T$, such that $\kappa(\mathbf{x})(t)=\mathbf{x}(t+\tau)$. We have $\kappa(\mathbf{x}^0)=\mathbf{x}(\tau)$. Let us compare $\kappa(\mathbf{x}^0)$ to the point $\Pi(\mathbf{x}^0)$, where $\mathbf{x}$ meets the plane $\{x_2=0\}$ for the first time. Note that $\kappa(\mathbf{x}^0)$ could as well, correspond to the nth crossing of this plane by $\mathbf{x}(t)$, with $n>1$. In fact this will not happen. In order to prove this, we observe that, near the bifurcation point, this solution moves nearly in the same direction as the solution of the linearized equation. The solution of the linearized equation for $\lambda=1/2$ lives in a plane which intersects the coordinate planes along the lines $x_i=x_j$. τ^* is the time needed by the solution of the linearized equation to get from one coordinate plane to the next one, τ^* is the sixth of the period for the linearized equation. The solutions on the local branch need nearly the same time. It follows from this that, near the bifurcation point, the periodic solutions cross at most one time each of the coordinate planes. Hence $\kappa(\mathbf{x}^0)=\Pi(\mathbf{x}^0)$. The last part of this reasoning is heuristic. A rigorous proof would require an exact evaluation of the return time. This is feasible but makes this proof of the same order of complication as the one we just did. We conclude from this that in the C^2 case, the local branch given by the Hopf bifurcation theorem is built up by solutions having the same symmetry properties as the solutions given by Theorem 2.1.

In the C^1 case, both our theorem and the local Hopf theorem fail to ensure uniqueness of solutions. But our method yields solutions with these symmetries, by construction.

3

In this section we prove a global bifurcation result using the Browder ejective fixed point theorem. To this end we construct a closed bounded convex set in $\mathbb{R}^+ \times \mathbb{R}^-$ which is invariant by the Poincaré operator Π. We need also a uniform bound of the first return time for the operator Π on K. From now on f is assumed to satisfy the hypotheses

> f satisfies (H) and f is monotone non-decreasing on $\mathbb{R}^+$.
> For every a, $a>0$, there exists $R=R(a)>0$ such that $f(R) \leqslant aR$. $\qquad (\tilde{\mathrm{H}})$

The last condition reads as $\liminf_{x \to \infty}(f(x)/x)=0$.

DEFINITION 3.1. For every $\lambda \in\,]0, 1/2[$, we define the convex set $K=K(\lambda)$ by

$$K=\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leqslant x_1,\ -x_2 \leqslant R(\lambda)\}.$$

LEMMA 3.1. *K is invariant by Π.*

Proof. Recall that we have

$$\dot{x}_1(t) = -\lambda x_1(t) - f(x_2(t))$$
$$\dot{x}_2(t) = -\lambda x_2(t) - f(x_3(t)$$
$$\dot{x}_3(t) = -\lambda x_3(t) - f(x_1(t)).$$

As in Section 2, we denote by τ the first time the component $x_2(t)$ of the solution $\mathbf{x}(t)$ vanishes. We choose $R(\lambda)$ according to $(\tilde{\mathrm{H}})$. Let $\mathbf{x}^0 = (x_1^0, x_2^0, 0)$ be such that $x_1^0 \geqslant 0 \geqslant x_2^0$ and $|x_1^0| \leqslant R(\lambda)$, $|x_2^0| \leqslant R(\lambda)$. Then we will show that $|x_i(t)| \leqslant R(\lambda)$, $0 \leqslant t \leqslant \tau$, for $i = 2, 3$. The conclusion will follow from the fact that $\Pi(x_1^0, x_2^0) = -(x_3(\tau), x_1(\tau))$. Integrating the second equation above from 0 to t, $t \leqslant \tau$ shows that $x_2(t)$ is negative and increasing, so, $|x_2(t)| \leqslant |x_2^0| \leqslant R(\lambda)$. From the first equation we get $x_1(t) = e^{-\lambda t} x_1^0 + \int_0^t e^{-\lambda(t-s)} f(x_2(s))\, ds$. Hence

$$|x_1(t)| \leqslant e^{-\lambda t} R(\lambda) + f(R(\lambda)) \frac{1 - e^{-\lambda t}}{\lambda} \leqslant R(\lambda).$$

On the other hand, $x_3(t) = \int_0^t e^{-\lambda(t-s)} f(x_1(s))\, ds$, which yields $|x_3(t)| \leqslant f(R(\lambda))((1 - e^{-\lambda t})/\lambda) \leqslant R(\lambda)$.

So, $0 \leqslant x_3(\tau)$, $-x_1(\tau) \leqslant R(\lambda)$. This proves that K is invariant by Π.

LEMMA 3.2. *Assume that f verifies hypothesis $(\tilde{\mathrm{H}})$. Then τ is uniformly bounded in K.*

Proof. We prove this in three steps.

Steps 1. We claim that for every $a > 0$, $a < R(\lambda)$, there exists $M = M(a)$, such that

$$x_1^0 \geqslant a > 0 \Rightarrow \tau \leqslant M.$$

By integration from 0 to t, we get

$$x_2(t) = e^{-\lambda t} x_2^0 - \int_0^t e^{-\lambda(t-s)} f(x_3(s))\, ds$$
$$= e^{-\lambda t} x_2^0 + \int_0^t e^{-\lambda(t-s)} f\left(\int_0^s e^{-\lambda(s-\sigma)} f(x_1(\sigma))\, d\sigma \right) ds.$$

We also have

$$x_1(\sigma) = e^{-\lambda\sigma} x_1^0 - \int_0^\sigma e^{-\lambda(\sigma-u)} f(x_2(u))\, du$$

so, $x_1(\sigma) \geqslant e^{-\lambda\sigma} x_1^0$, since $x_2 \leqslant 0$.

Hence

$$x_2(t) \geqslant e^{-\lambda t} x_2^0 + \int_0^t e^{-\lambda(t-s)} f\left(\int_0^s \left(e^{-\lambda(s+\sigma)} f(e^{-\lambda\sigma} x_1^0)\right) d\sigma\right) ds.$$

This inequality yields a more tractable one by using the fact that $f(x) \geqslant kx$ for some $k > 0$, and $x \in [0, R(\lambda)]$.

We obtain

$$x_2(t) \geqslant e^{-\lambda t} x_2^0 + k^2 \left(\int_0^t e^{-\lambda t} \left(\int_0^s d\sigma\right) ds\right) x_1^0.$$

Hence $x_2(t) \geqslant e^{-\lambda t}[x_2^0 + (1/2) k^2 x_1^0 t^2]$.

Thus, $x_2(\tau) = 0 \Rightarrow x_2^0 + (1/2) k^2 x_1^0 \tau^2 \leqslant 0$, that is,

$$\tau \leqslant \left(\frac{2\,|x_2^0|}{k^2\,|x_1^0|}\right)^{1/2}.$$

We can take

$$M(a) = \left(\frac{2R(\lambda)}{k^2 a}\right)^{1/2}.$$

Step 2. In this step we show that τ remains bounded as the initial data $\mathbf{x}^0$ tend to zero. We proceed by contradiction Assume that there exists a sequence of initial data $\mathbf{x}_n^0$ converging to zero such that $\tau(x_n^0) \to +\infty$. We set $A' = -\lambda I - A$, then

$$\mathbf{x}_n(t) = e^{A't} \mathbf{x}_n^0 + \int_0^t e^{A'(t-s)} \mathbf{G}(\mathbf{x}_n(s))\, ds.$$

Dividing both sides by $\|\mathbf{x}_n^0\|$, we obtain

$$\frac{\mathbf{x}_n(t)}{\|\mathbf{x}_n^0\|} = e^{A't} \frac{\mathbf{x}_n^0}{\|\mathbf{x}_n^0\|} + \int_0^t e^{A'(t-s)} \frac{G(\mathbf{x}_n(s))}{\|\mathbf{x}_n^0\|}\, ds.$$

Let $y_n(t) = \mathbf{x}_n(t)/\|\mathbf{x}_n^0\|$. From the above equation and its differentiated form, we see that the sequences $y_n(t)$ and $(d/dt)\, y_n(t)$ are bounded in $C([0, T], \mathbb{R}^2)$ for all $T > 0$.

By the Ascoli–Arzela theorem there exists a subsequence $y_{n_i}(t)$ of $y_n(t)$ which converges on $\mathbb{R}^+$ to some function $y(t)$.

$y(t)$ satisfies the equation $(d/dt)\, y(t) = A' y(t)$ and $y(t)$ is $\neq 0$ since

$$\|y_n(0)\| = 1 \qquad \text{and} \qquad \|y(0)\| = \lim_{n \to \infty} \|y_n(0)\|.$$

Moreover we have $y_1(t) \geqslant 0$, $y_2(t)$, and $y_3(t) \leqslant 0$ for all t. Let us prove that this is impossible.

Recall that for $\lambda < 1/2$, $\mathbf{A}'(\lambda)$ has three eigenvalues say λ_0, λ_1, λ_2. λ_0 is real and corresponds to an eigenvector with positive components; λ_1 and λ_2 are complex numbers with a positive real part. Let us write $y(t) = ae^{\lambda_0 t}V + be^{\lambda_1 t}V_1 + ce^{\lambda_2 t}V_2$. Then we see that since the components of y keep a constant sign, the oscillatory part of the solution [which would dominate if present] has to vanish: $b = c = 0$ and $y(t) = ae^{\lambda_0 t}V$.

Hence all components of $y(t)$ are of the same sign. Since $y_1(t) \geqslant 0$ and $y_2(t) \leqslant 0$, $y_3(t) \leqslant 0$, this implies that $y_1(t) = y_2(t) = y_3(t) = 0$. This yields a contradiction.

Steps 3. We now consider the case where $|x_2^0| \geqslant a > 0$. Assume that there exists a sequence $(\mathbf{x}^n(0))$ such that the $\tau(\mathbf{x}^n(0)) \to +\infty$ with $-R(\lambda) \leqslant x_2^n(0) \leqslant -a$ where a is >0. Then Step 1 implies that $x_1^n(0) \to 0$ and we may assume that $x_2^n(0)$ converges to some point say x_2. Let us set $\mathbf{x} = (0, x_2, 0)$, we claim that $\tau(\mathbf{x}) = +\infty$. In fact suppose that $\tau(\mathbf{x}) < +\infty$, then there exists $\tau' > \tau(\mathbf{x})$ such that $x_2(\tau', \mathbf{x}) > 0$ hence by continuity of the solutions with respect tot the initial data, there exists a neighborhood V of $\mathbf{x}$ such that $\forall \mathbf{x}^0 \in V$, $x_2(\tau', \mathbf{x}^0) > 0$. This implies the existence of n_0 such that for $n \geqslant n_0$ we have $\mathbf{x}^n(0) \in V$, hence $x_2(\tau', \mathbf{x}^n(0)) > 0$ which means that $\tau(\mathbf{x}^n(0)) \leqslant \tau'$. This contradicts the hypothesis $\tau(\mathbf{x}^n(0)) \to +\infty$. On the other hand by Lemma 1.4 we know that the solutions corresponding to initial values of the form $\mathbf{x} = (0, x, 0)$, intersect the coordinate planes after a finite time, that is, $\tau(\mathbf{x}) < \infty$. This contradiction shows that τ is bounded in the compact set $K_3 = \{(x_1, x_2)/0 \leqslant x_1 \leqslant R(\lambda);\ -R(\lambda) \leqslant x_2 \leqslant -a;\ a > 0\}$.

Let us summarize: in Step 1 we considered the compact set $K_1 = \{(x_1, x_2)/-R(\lambda) \leqslant x_2 \leqslant 0;\ 0 < a \leqslant x_1 \leqslant R(\lambda)\}$. In Step 2 we introduced the compact set $K_2 = \{\mathbf{x} = (x_1, x_2)/\|\mathbf{x}\| \leqslant \varepsilon\}$. ε and a can be chosen at our convenience, $a < \varepsilon$. On each of these compact sets we got a uniform bound for τ. Hence we have a uniform bound for τ on K since $K = K_1 \cup K_2 \cup K_3$. This completes the proof of Lemma 3.2. ∎

As an immediate consequence of Lemma 3.1, we have the

COROLLARY 3.1. *The first return time is bounded by a constant, say T, independent of $\mathbf{x}^0$ in K.*

LEMMA 3.3. *For every $\varepsilon' > 0$ there exists $\varepsilon > 0$ such that $\|\mathbf{x}^0\| \leqslant \varepsilon \Rightarrow \|\mathbf{x}(t, \mathbf{x}^0)\| \leqslant \varepsilon'$ for $0 \leqslant t \leqslant T$.*

Proof. Consider the equation $\dot{\mathbf{x}}(t) = F(\mathbf{x}(t))$. We can find some constant C much that $(d/dt)\, \|\mathbf{x}(t)\| \leqslant C\, \|\mathbf{x}(t)\|$ in the compact K, since F is locally Lipschitz continuous.

Hence $\|\mathbf{x}(t)\| \leqslant e^{Ct}\, \|\mathbf{x}^0\|$; thus one can take $\varepsilon = \varepsilon' e^{CT}$. ∎

DEFINITION 3.2. A point $e \in \mathbb{R}^+ \times \mathbb{R}^-$ is said to be an ejective point of Π if there exists an open neignborhood U of e such tha for every $\mathbf{x}^0$, $\mathbf{x}^0 \in U$, we have $\Pi^k \mathbf{x}^0 \notin U$ for some $k > 0$.

LEMMA 3.4. *The zero solution is an ejective fixed point of the Poincaré operator.*

Proof. We proceed by contradiction. Assume that for every open neighborhood U of zero there exists $\mathbf{x}^0 \in U$ such that $\Pi^k \mathbf{x}^0 \in U$ for all $k \geqslant 0$. Given $\varepsilon' > 0$ and the corresponding ε in Lemma 3.2, we set $U = B(0, \varepsilon)$ and we choose $\mathbf{x}^0$ such that $\Pi^k \mathbf{x}^0 \in B(0, \varepsilon)$. Applying Lemma 3.2 to $\Pi^k \mathbf{x}^0$ we get $\|x(t, \Pi^k \mathbf{x}^0)\| \leqslant \varepsilon'$ for $0 \leqslant t \leqslant T$. Setting $\mathbf{x}(t_k, \mathbf{x}^0) = \Pi^k \mathbf{x}^0$ we have $\|\mathbf{x}(t, \mathbf{x}^0)\| \leqslant \varepsilon'$ for $t_k \leqslant t \leqslant t_k + T$. In particular $\|\mathbf{x}(t, \mathbf{x}^0)\| \leqslant \varepsilon'$ for $t_k \leqslant t \leqslant t_{k+1}$ since $t_{k+1} - t_k \leqslant T$. Thus $\|\mathbf{x}(t, \mathbf{x}^0)\| \leqslant \varepsilon'$ for all $t \geqslant 0$. On the other hand the system is hyperbolic for $\lambda \neq 1/2$. Hence one can find $\varepsilon' > 0$ such that $\|\mathbf{x}(t, \mathbf{x}^0)\| < \varepsilon'$ for all $t \geqslant 0$, implies that $\mathbf{x}(t, \mathbf{x}^0)$ belongs to the stable manifold passing through zero. This means that $\mathbf{x}(t, \mathbf{x}^0)$ tends to 0 as t tends to $+\infty$ and the orbit of $\mathbf{x}(t, \mathbf{x}^0)$ s tangent to the stable manifold corresponding to the linearized equation. For $\lambda < 1/2$, this manifold is associated to the unique real eigenvalue whose eigenvectors have all their components of the same sign. This means that the components of $\mathbf{x}(t, \mathbf{x}^0)$ are all of the same sign for t large enough. Hence $\sigma(\mathbf{x}(t, \mathbf{x}^0)) > 0$ for t large enough. But we know that $\sigma(\mathbf{x}(t_k, \mathbf{x}^0))$ is < 0 since $\mathbf{x}(t_k, \mathbf{x}^0) = \Pi^k \mathbf{x}^0$ belongs to the plane $\{(x_1, x_2, 0) \mid x_1 > 0 > x_2\}$. This yields a contradiction for k large enough. Hence, zero is an ejective point for Π. ▮

Let us now recall the Browder ejective fixed point theorem:

THEOREM [4]. *If K is a closed, bounded, convex, infinite-dimensional set in a Banach space X, A: $K \setminus \{e\} \to K$ is completely continuous, and $e \in K$ is an ejective point of A, then there is a fixed point of A in $K \setminus \{e\}$. If K is finite dimensional the same conclusion holds, if in addition to being ejective, e is an extreme point of K.*

We can now complete the proof of Theorem 3.1: we know by Lemma 3.3 that 0 is an ejective fixed point for Π in K (of Definition 3.1). K is a closed, bounded convex subset of $\mathbb{R}^3$ having zero as an extreme point. All the conditions of Browder's ejective fixed point theorem are fulfilled, yielding the desired conclusions. ▮

Relation with Retarded Differential Equations

Let $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ be a periodic solution of (0.1) associated with a fixed point of Π. Let T be the period of $\mathbf{x}$. Set $\tau = T/6$. We know from the preceding results that $(x_1(\tau), x_2(\tau), x_3(\tau)) = -(x_2(0), x_3(0), x_1(0))$.

Hence we have $x_1(t+\tau)=-x_2(t)$, $x_2(t+\tau)=-x_3(t)$, and $x_3(t+\tau)=-x_1(t)$. Thus $x_1(t)=-x_2(t-\tau)$, $x_2(t)=-x_3(t-\tau)$, and $x_3(t)=-x_1(t-\tau)$.

Finally we get

$$x_1(t)=x_3(t-2\tau), \qquad x_2(t)=x_1(t-2\tau), \qquad \text{and} \qquad x_3(t)=x_2(t-2\tau).$$

We deduce from this that $\dot{x}_1(t)=-\lambda x_1(t)-f(x_2(t))=-\lambda x_1(t)-f(x_1(t-2\tau))$. This means that the first component of a periodic solution of systems (0.1) obtained from a fixed point of Π satisfies the delay differential equation

$$\dot{x}(t)=-\lambda x(t)-f(x(t-r)),$$

where $r=2\tau$.

It is a "slowly oscillating" periodic solution according to the terminology we recalled in the Introduction. Delay differential equations as above have been notably studied by K. P. Hadeler nd J. Tomiuk, see [3]. More precisely, these authors considered a 2-parameter family of emuations

$$\frac{dx}{dt}=-px(t)-qf(x(t-1)), \tag{3.1}$$

normalizing the delay as well as $f'(0)$ to 1. They proved the existence of a slowly oscillating periodic solution for each (p, q) in a region R defined as $R=\{(p, q): p, q>0,\ q\geqslant\alpha(p)\}$. Here $\alpha(p)$ denotes, for each p, the first value of the parameter q for which the characteristic equation associated with the linearization of (3.1) around zero has an imaginary root, see [3]. In the frame of Eq. (3.1), our result can be stated as follows: for each λ, $0<\lambda<1/2$, there exists $(p, q)\in R$, $p=\lambda q$, such that Eq. (3.1) for this choice of (p, q) has at least one non-trivial slowly oscillating periodic solution of period 3.

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