DELAY DIFFERENTIAL SYSTEMS

ASYMPTOTICALLY EQUIVALENT TO ORDINARY DIFFERENTIAL SYSTEMS

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1. PRESENTATION

We consider equations such as:

$$\frac{dx}{dt} = L(t,x_t) \; ; \tag{L}$$

$L(t,\phi)$ being defined on $[t_o,+\infty) \times C([-r,0],\mathbb{R}^n)$, linear in ϕ and asymptotically autonomous:

i.e.
$$\lim_{t\to+\infty} L(t,\phi) = L_\infty(\phi) \, . \tag{L_∞}$$

As usual, x_t will denote the function defined on $[-r,0]$ by:

$$x_t(\theta) = x(t+\theta), \quad -r \leq \theta \leq 0 \, .$$

Our general problem is to find asymptotic formulae for the solutions of (L). Let us start with some examples:

$$\frac{dx}{dt} = p(t) \cdot \big(x(t) - x(t-1)\big), \tag{1}$$

with a growth condition on $p(t)$, such as:

$$p \text{ in } L^2(t_o,+\infty) \quad \text{or} \quad \limsup_{s\to+\infty} |p(s)| < 1$$

The asymptotic constancy of the solution of (1) has been proved by many authors: ([11],[4],[7]).

Here, the asymptotic equation is: $dx/dt = 0$

$$\frac{dx}{dt} = a \cdot x\big(t - r(t)\big) . \tag{2}$$

Numerous equations can be reduced to this compact form. It is therefore important to look for asymptotic behaviour of its solutions. With r in $L^p(t_o,+\infty)$, $r(t) \to 0$, $t \to \infty$, the first results are due to K.L. Cooke ([5]). Here, the asymptotic equation is: $dx/dt = a \cdot x(t)$, and, if r is in $L^1(t_o,+\infty)$, then: K.L. Cooke proved that the solutions of (2) behave asymptotically in the same way that the solutions of its asymptotic equation do.

If x is a solution of (2), there exists C such that:

$$x(t) - C \cdot e^{at} = o(e^{at})$$

$$\frac{dx}{dt} = \big(\lambda + a(t)\big)x(t) + b(t)x(t - r) \tag{3}$$

$$\frac{dx}{dt} = \big(\Lambda + A(t)\big) \cdot x(t) + B(t) \cdot x(t - r) \tag{4}$$

(under a vectorial form, where Λ is a diagonal matrix). Under its vectorial form, this equation has been proposed by J.R. Haddock and R.J. Sacker ([8]) as a model in a study aiming at extending results by P. Hartman and A. Wintner [9] for ordinary differential equations.

$A(t)$ and $B(t)$ are perturbative terms, for example: A,B in $L^2(t_o,+\infty)$. In [8], Haddock and Sacker proved an asymptotic result for the scalar case and stated a conjecture for the vectorial case.

At the conference Equadiff 82 [1] we presented a general method which encompasses these examples, and notably we developed the Haddock and Sacker conjecture.

Here, we would like to touch on another aspect of our work. We note that, even if the asymptotic behaviour of the systems under consideration does not reduce to the one of the asymptotic equations, nevertheless the flow is asymptotically degenerate: the information about the limiting behaviour of a solution depends only on a finite number of parameters.

This leads us to the following problem: in what way could the asymptotic behaviour be described with an ordinary differential system, conveniently associated to (L)?

2. PRELIMINARIES

We start with a simple observation: considering the system (L), we can associate to (L) a number of ordinary systems. We have only to choose a continuous map $i : \mathbb{R}^n \to C([-r,0],\mathbb{R}^n)$, and define:

$$\frac{dx}{dt} = L\big(t,i(x(t))\big)\,. \qquad \big(L(i)\big)$$

But obviously not all of these equations will be of interest in view of the asymptotic behaviour. More precisely, there is no relation in general between (L) and $\big(L(i)\big)$. We note however the following:

PROPOSITION 1. Let $U(t,s)$ denote the resolvent of (L), and suppose that:

$$\|U(t,s)\| \leq 1,$$

and let i be such that: $\|i\| \leq 1$. Then: $u(i)(t,s)$ being the resolvent of $L(i)$, we have:

$$|u(i)(t,s)| \leq 1, \quad t \geq s.$$

REMARK 1. To prove this result, we observe that the solutions of $(L(i))$ can be obtained as limits of sequences of solutions of (L), in such a way that if x_o is a datum for $(L(i))$ all the approximations y_N verify:

$$\| y_N\| \leq |x_o| .$$

To simplify, choose: $i(x)$ = the constant x, denote by: $L_o(t,x) = L(t,i(x))$, and, by $u(t,s)$ the resolvent of (L_o).

From the stability of (L_o) we can obtain some results for (L). Precisely, using $u(t,s)$, we can compare (L) to perturbations such as

$$\frac{dx}{dt} = L(t,x_t) + f(t) \qquad (L_f)$$

PROPOSITION 2. ([1],[2]). Suppose $\alpha > 0$ is such that:

$$r : \Big(\limsup_{t \to +\infty} \|L(t,\cdot)\|\Big)^2 \cdot \limsup_{\substack{t \leq s \\ t \to +\infty}} \|u(t,s)\| \cdot \frac{e^{2\alpha r}}{\alpha} < 1 .$$

Then: for any f such that: $\|f(t)\| = O(e^{-\alpha t})$, the systems (L_f) and (L) are L^1-asymptotically equivalent, i.e.: for every solution x of (L_f) there exists a solution y of (L) such that:

$$\int_{t_o}^{+\infty} \|x(t) - y(t)\| dt < +\infty$$

3. THE MAIN RESULT

Up-to-now we have not answered the question stated at the end of I. In fact, this question was solved, in a completely different approach, by Ryabov [10] who introduced the notions of "special solutions" and showed their existence and "completeness" ([10],[7]) in certain systems (L). Later on, we will give more details about that. Let us say that the interest of such results in the search for asymptotic behaviour has been notably underlined by R.D. Driver in [6].

There is no question of ordinary differential equations in these author's views, but we will see that it is not difficult to pass from the frame of "special solutions" to the one of "asymptotically equivalent" ordinary differential systems.

We now state the result under this last form:

THEOREM. Assume

$$(L_1) : \|L(t,\cdot)\| \leq K, \quad t \in \mathbb{R};$$

$$(L_2) : K \cdot r \cdot e < 1$$

(from (L_2) it follows that there is a unique μ in $(0, 1/r)$ such that: $\mu = K \cdot e^{\mu r}$).

Then there exists an ordinary differential system:

$$\frac{dx}{dt} = \ell\big(t, x(t)\big), \tag{ℓ}$$

the solutions of which are solutions of (L), (ℓ) verifying (L_3): $\|\ell(t,\cdot)\| \leq \mu$.

Moreover, the system (ℓ) is, within $\mathbb{R}^n$-isomorphisms, uniquely determined by the conditions (L_1), (L_2) and (L_3).

Finally, if $Y(t)$ is a fundamental system of solutions of (ℓ) (with $t_o = 0$), then for every solution x of (L) there exists a unique c in $\mathbb{R}^n$, such that: $x(t) = Y(t) \cdot \big(c + o(1)\big)$.

To prove the theorem, we only have to come back to the original result by Ryabov.

We first recall what Ryabov calls a "special solution":

DEFINITION ([6]). A "special solution" is a solution of (L), defined on $\mathbb{R}$, growing at most exponentially, with an exponent not greater than $1/r$.

Ryabov then proved:

LEMMA ([10],[6]). Assume (L_1), (L_2) and (L_3). Then, for each (t_o, y_o) in $\mathbb{R} \times \mathbb{R}^n$, there exists a unique special solution passing through y_o at t_o. The set of the special solutions is an n-dimensional space.

Each special solution $y(t)$ satisfies an estimate:

$$\|y(t)\| \leq \|y_o\| \cdot e^{\mu|t-t_o|}, \qquad t \in \mathbb{R}.$$

REMARK 2. The first part of this lemma means that such a system of solutions is complete.

All we have to do in order to prove the theorem is:

(i) observe that a complete family of special solutions is associated to an ordinary equation in $\mathbb{R}^n$;

(ii) that there is uniqueness within isomorphism;

(iii) prove the asymptotic formula (end of THM).

We will prove (i), skip (ii) and go very fast on (iii). To prove (i), let x be a special solution. It can be expressed as:

$$x(t) = x(t_o) + \int_{t_o}^{t} L(s,x_s)ds . \tag{5}$$

Because of the uniqueness property stated in the lemma, we can see that, for each θ in $[-r,0]$, s in $\mathbb{R}$, $x(s+\theta)$ is uniquely determined in terms of $x(s)$, so that $x(s) \to x(s+\theta)$ defines a map $G(s,\theta)$.

Because of the lemma, we have:

$$\|G(s,\theta)\| \leq e^{\mu r} .$$

Using G, (5) can be written as:

$$x(t) = x(t_o) + \int_{r_o}^{t} L\big(s,G(s,\cdot)\cdot x(s)\big)ds .$$

Let: $\ell(s,x) = L\big(s,G(s,\cdot)\cdot x\big)$. We then have: $dx/dt = \ell\big(t,x(t)\big)$, which yields the first part of the theorem. Moreover:

$\|\ell(s,\cdot)\| \leq K \cdot e^{\mu r} = \mu$; so, we get (L_3). The last part of the theorem has been proved by R.D. Driver [6] using "special solutions."

We transform the equation using the resolvent $Y(t,s)$ of (ℓ) (in fact, we use: $Y(t) = Y(t,0) : x(t) = Y(t) \cdot z(t)$).

Using the Gronwall-type inequality ([1],[3]) we can see that: $(d/dt)z$ is in $L^1(t_o, +\infty)$, so that z has a limit at $+\infty$, and for each c in $\mathbb{R}^n$, there exists a solution z (and so a solution x) such that: $z(t) \to c$, $t \to +\infty$.

4. CONCLUDING REMARKS

Our theorem is a perturbative result: for $r = 0$, (L) is an o.d.e. in $\mathbb{R}^n$. For $r > 0$ small (see (L_2)), there is still an o.d.e. subsystem of (L) — an o.d.e. in $\mathbb{R}^n$, that contains the information on asymptotic behaviour. Why now do we consider a formulation in terms of o.d.e.'s? In what way could this concept be more interesting than the one of special solutions? The answer to these questions can only be partial. In ([2]), we combined the o.d.e. formulation with results on asymptotic integration of o.d.e's to get asymptotic formulae for functional differential systems.

Another interesting feature is that (ℓ) provides us with a natural simple adjoint equation in $(\mathbb{R}^n)^*$, which in fact can be used to describe the limiting behaviour of the solutions of (L). Precisely, there exists a fundamental solution Y^* of (ℓ^*) such that: $c = \lim_{t\to+\infty} \langle Y^*(t), x(t) \rangle$ (where c, x are as in the theorem).

On the other hand, the notion of a "subsystem" is still "theoretical," it needs much more work to be really useful, and notably the following question can be set: Is it possible to get such subsystems without the intermediary of special solutions?

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