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PERIODIC SOLUTIONS FOR RETARDED DIFFERENTIAL SYSTEMS CLOSE TO ORDINARY ONES

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INTRODUCTION

IN THIS paper we will consider the retarded differential system (abbreviated as RDS):

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t-r)) \tag{0.1}$$

where we assume that f is a smooth function from $\mathbb{R}^2$ into $\mathbb{R}^2$, and

$$(\mathrm{H}_1) \qquad f(0) = 0; \qquad Df(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The object of our investigations is to obtain existence of periodic solutions when r is a small positive real number.

For r close to 0, the system (0.1) is a perturbation of an ordinary differential system.

On the other hand, using the variable $y(t) = x(tr)$ instead of x, the RDS (0.1) can be read as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = rf(y(t-1)). \tag{0.2}$$

We observe that for a T-periodic solution ($T > 0$) $x(t)$ of (0.1) the corresponding periodic solution $y(t)$ of (0.2) has a period T/r which is close to $+\infty$ when r is close to 0. It is a bifurcating phenomenon at infinity. Therefore, Hopf bifurcation theory does not apply.

The purpose of this work is to construct a perturbation method to get existence of periodic solutions of the system (0.1) in the case where r is small enough.

Since, for r small enough, the system is a perturbation of an ordinary differential system, we transform it into a perturbed parametric ordinary differential system and we construct a fixed point problem. We underline that our technic is essentially based on some stability property of the origin of (0.1) when $r = 0$. Precisely we will assume the origin of (0.1) is 3-asymptotically stable for $r = 0$ [5].

In Section 1, we recall the concept of h-asymptotic stability and Hopf bifurcation developed in [2, 5].

In Section 2, we present our perturbation method and we show that the RDS (0.1) has at least one period solution.

1. PRELIMINARIES

We recall the framework for h-asymptotic stability and Hopf bifurcation [2, 4, 5].

Consider the system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_1 = \alpha(\mu) x_1 - \beta(\mu) x_2 + P(\mu, x_1, x_2) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t} x_2 = \alpha(\mu) x_2 - \beta(\mu) x_1 + Q(\mu, x_1, x_2) \end{cases} \tag{1.1}$$

where

$$\alpha(\mu), \beta(\mu) \in \mathcal{C}^{k+1}(]-\bar{\mu}, \bar{\mu}[, \mathbb{R})$$

$$P, Q \in \mathcal{C}^{k+1}(]-\bar{\mu}, \bar{\mu}[\times B^2(a), \mathbb{R}) \qquad \text{with } k \text{ an integer, } k \geq 3,$$

such that

$$\alpha(0) = 0, \qquad \beta(0) = 1 \qquad \text{and} \quad \frac{\mathrm{d}\alpha}{\mathrm{d}\mu}(0) \neq 0$$

$$P(\mu, 0, 0) = Q(\mu, 0, 0) = 0 \qquad \text{and} \qquad D_x P(\mu, 0, 0) = D_x Q(\mu, 0, 0) = 0.$$

Introducing polar coordinates:

$$x_1 = p \cos\theta, \qquad x_2 = p \sin\theta$$

we have

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}t} = \alpha(\mu)\rho + P^*(\mu, \rho, \theta)\cos\theta + Q^*(\mu, \rho, \theta)\sin\theta \\[2ex] \rho \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \beta(\mu)\rho + Q^*(\mu, \rho, \theta)\cos\theta - P^*(\mu, \rho, \theta)\sin\theta \end{cases} \tag{1.2}$$

where

$$P^*(\mu, \rho, \theta) = P(\mu, \rho\cos\theta, \rho\sin\theta)$$

$$Q^*(\mu, \rho, \theta) = Q(\mu, \rho\cos\theta, \rho\sin\theta)$$

$$\begin{cases} W(\mu, \rho, \theta) = \beta(\mu) + (Q^*(\mu, \rho, \theta)\cos\theta - P^*(\mu, \rho, \theta)\sin\theta)/\rho \qquad \text{if } \rho \neq 0 \\ W(\mu, 0, \theta) = \beta(\mu). \end{cases}$$

For every $\rho_0 \in [0, a[$ and $\theta_0 \in \mathbb{R}$, the orbit of (1.1) passing through (ρ_0, θ_0) will be represented by means of the noncontinuable solution $\rho(\theta, \mu, \theta_0, \rho_0)$ of the problem:

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}\theta} = \mathcal{R}(\mu, \rho, \theta) \\[2ex] \rho(\theta_0, \mu) = \rho_0 \end{cases} \tag{1.3}$$

where

$$\mathcal{R}(\mu, \rho, \theta) = \frac{\alpha(\mu)\rho + P^*(\mu, \rho, \theta)\cos\theta + Q^*(\mu, \rho, \theta)\sin\theta}{W(\mu, \rho, \theta)}. \tag{1.4}$$

When the function $\rho(\mu, \theta, \rho_0, \theta_0)$ has been determined, the complete knowledge of the solutions of (1.2) will be obtained by integration of the following equations:

$$\begin{cases} \dfrac{d\theta}{dt} = W[\mu, \rho(\mu, \theta, \rho_0, \theta_0), \theta] \\ (\mu, \rho, \theta) \in]-\bar{\mu}, \bar{\mu}[\times [0, a[\times \mathbb{R}. \end{cases} \tag{1.5}$$

Since $\alpha(0) = 0$, it is easily seen by (1.3) and (1.4) that if $\bar{a} \in [0, a[$ and μ are sufficiently small for any $\mu \in]-\bar{\mu}, \bar{\mu}[$ and $c \in [0, a[$ the solution of (1.5) exists in $[0, 2\pi]$. This solution will be denoted by $\rho(\mu, \theta, c)$.

Definition 1.1 [5]. The function $V(\mu, c) = \rho(\mu, 2\pi, c) - c$ is called a displacement function for (1.1).

Remark 1.1. Since $\mathcal{R}$ is $\mathcal{C}^k$, we have:

$\rho(\mu, \theta, c) = \sum_{i=1}^{k} U_i(\mu, \theta)c^i + \Phi(\mu, \theta, c)$; where Φ is of order $>k$

(i.e. $\Phi(\mu, \theta, c) = o(c^k)$)

U_i, Φ are $\mathcal{C}^k$ for $1 \leq i \leq k$ and $U_1(\mu, 0) = 1$,

$U_2(\mu, 0) = \cdots = U_k(\mu, 0) = \Phi(\mu, 0, c) = 0.$

For $\mu = 0$, the system (1.1) can be written in the form:

$$\begin{cases} \dfrac{d}{dt}x_1 = -x_2 + X(x_1, x_2) \\ \dfrac{d}{dt}x_2 = x_1 + Y(x_1, x_2) \end{cases} \tag{1.6}$$

where:

$$X(x_1, x_2) = P(0, x_1, x_2) \quad \text{and} \quad Y(x_1, x_2) = Q(0, x_1, x_2).$$

Definition 1.2 [5]. Let h be an integer; $h \in \{2, \ldots, k\}$. The solution $x_1 = x_2 = 0$ of (1.6) is said to be h-asymptotically stable (resp. h-completely unstable) if:

(i) for every $\tau, \zeta \in \mathcal{C}[B^2(a), \mathbb{R})$ of order greater than h; the solution $x_1 = x_2 = 0$ of the system

$$\begin{cases} \dfrac{d}{dt}x_1 = -x_2 + X_2(x_1, x_2) + \cdots + X_h(x_1, x_2) + \tau(x_1, x_2) \\ \dfrac{d}{dt}x_2 = x_1 + Y_2(x_1, x_2) + \cdots + Y_h(x_1, x_2) + \zeta(x_1, x_2) \end{cases} \tag{1.7}$$

is asymptotically stable [resp. completely unstable];

(ii) property (i) is not satisfied when h is replaced by any integer $m \in \{2, \ldots, h-1\}$.

THEOREM 1.1 [5]. Let h be an integer, $2 \le h \le k$. The following propositions are equivalent: (1) the solution $x_1 = x_2 = 0$ of (1.6) is h-asymptotically stable [resp. h-completely unstable]; (2) one has

$$\frac{\partial^i V}{\partial c^i}(0,0) = 0 \text{ for } 1 \le i \le h-1 \qquad \text{and} \qquad \frac{\partial^h V}{\partial c^h}(0,0) < 0 \qquad [\text{resp.} > 0].$$

In addition if either proposition (1) or (2) holds, h is odd.

THEOREM 1.2 [5]. There exist ε near 0, $\varepsilon > 0$ and a function μ in $\mathcal{C}^{k-1}([0, \varepsilon[, \mathbb{R})$ with $\mu(0) = (\mathrm{d}\mu/\mathrm{d}c)(0) = 0$ and $\sup\{|\mu(c)|, c \in [0, \varepsilon[\} = \bar{\varepsilon} < \bar{\mu}$ such that for any $c \in [0, \varepsilon[$, $\mu \in]-\bar{\varepsilon}, \bar{\varepsilon}[$ the orbit of (1.1) passing through $(0, c)$ is closed if and only if $\mu = \mu(c)$.

Remark 1.2 [5]. Theorem 1.2 is the $\mathcal{C}^{k+1}$ version of $\mathbb{R}^2$ of the local Hopf bifurcation theorem.

LEMMA 1.1. If the origin of (1.6) is 3-asymptotically stable then the amplitude of the bifurcating periodic solution of (1.1) [for μ close to $\mu = 0$] is of order $\sqrt{\mu}$. Furthermore the bifurcation is supercritical if $(\mathrm{d}\alpha/\mathrm{d}\mu)(0) > 0$.

Proof. If the origin of (1.6) is 3-asymptotically stable then we have:

$$\frac{\partial V}{\partial c}(0,0) = \frac{\partial^2 V}{\partial c^2}(0,0) = 0 \qquad \text{and} \qquad \frac{\partial^3 V}{\partial c^3}(0,0) < 0,$$

where $V(\mu, c)$ is the displacement function of (1.1). Moreover the bifurcating function: $c \to \mu(c)$ satifies:

$$\mu(0) = \frac{\mathrm{d}\mu}{\mathrm{d}c}(0) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2\mu}{\mathrm{d}c^2}(0) = \frac{1}{3}\left[\frac{\partial^3 V}{\partial c^3}(0,0) \Big/ \frac{\partial \hat{V}}{\partial \mu}(0,0)\right]$$

with $\hat{V}(\mu, c) = (V(\mu, c))/c$ if $c \ne 0$ $\hat{V}(\mu, 0) = U_1(\mu, 2\pi) - 1$; $U_1(\mu, 2\pi)$ is given by remark (1.1) and we have

$$U_1(\mu, 2\pi) = \exp\left(2\pi \frac{\mathrm{d}\alpha(\mu)/\mathrm{d}\mu}{\beta(\mu)}\right).$$

Then

$$\frac{\partial \hat{V}}{\partial \mu}(0,0) = 2\pi \frac{\mathrm{d}\alpha}{\partial \mu}(0) > 0.$$

Also

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}c^2}(0) > 0.$$

Furthermore, from the local Hopf bifurcation theorem [theorem 1.2], $\mu(c)$ is $(k-1)$-times continuously differentiable. So:

$$\mu(c) = \mu(0) + \frac{\mathrm{d}\mu}{\mathrm{d}c}(0) \cdot c + \frac{1}{2} \cdot \frac{\mathrm{d}^2\mu}{\mathrm{d}c^2}(0) \cdot c^2 + o(c^2),$$

then:

$$\mu(c) \simeq \frac{1}{2}\frac{\mathrm{d}^2\mu}{\mathrm{d}c^2}(0) \cdot c^2 \qquad \text{and} \qquad c \simeq \sqrt{\frac{2\mu(c)}{\mathrm{d}^2\mu(0)/\mathrm{d}^2 c}}.$$

This shows that the amplitude of the bifurcating periodic solution of (1.1) [for μ close to $\mu = 0$] is of the order $\sqrt{\mu}$. Since $(d^2\mu/dc^2)(0) > 0$, from [2], we deduce that the bifurcation occurs for $\mu > 0$. ■

2. MAIN RESULT

In this section we develop our existence result. We proceed in the following way. We transform the system (0.1) into a perturbed parametric ordinary differential system and we construct a fixed point problem in the neighborhood of the bifurcating solutions of the ODS.

PROPOSITION 2.1. Under (H_1), (0.1) can be written in the form:

$$\frac{d}{dt}x(t) = g(r, x(t)) + H(x_t) \tag{2.1}$$

where:

$$g(r, x) = [I + rDf(x)]^{-1}f(x) \tag{2.2}$$

and $H(\phi)$ is defined as the difference:

$$H(\phi) = f(\phi(-r)) - g(r, \phi(0)). \tag{2.3}$$

Moreover, let $T > 0$ and $R > 0$ be fixed. Suppose

$\phi = x_t$, for a solution of (0.1) such that

$\|x_0\| \leq R$ and some time t,

(i) $3r \leq t \leq T$, then: $H(\phi) = o(r^2)$ (uniformly with respect to T and R).

(ii) If, on the other hand, $0 \leq t \leq 3r$, then: $H(\phi) = O(r^{3/2})$ (once more, uniformly with respect to T and R).

Before proving proposition 2.1, we look for a while at the ordinary differential system

$$\frac{d}{dt}y = g(r, y) \tag{2.4}$$

where $g(r, y)$ is defined by formula (2.2).

From now on, we will assume that:

(H_2) for $r = 0$, the origin $y = 0$ of (2.4) is 3-asymptotically stable.

PROPOSITION 2.2. Under the assumptions (H_1) and (H_2) the system (2.4) has a family of periodic solutions parametrized by r, for r close to 0, of amplitude of order $\sqrt{r}$ and of period close to 2π.

Proof. g is defined from $\mathbb{R} \times \mathbb{R}^2$ into $\mathbb{R}^2$ and satisfies:

(i) $g(r, 0) = 0$, $\forall\, r \in \mathbb{R}_+^*$

(ii) $D_y g(r, 0) = [I - rDf(0)]^{-1}Df(0)$.

Notice that $D_y g(r, 0)$ has a complex pair of conjugate eigenvalues $\alpha(r) \pm i\beta(r)$ with $\alpha(r) = r/(1 + r^2)$ and $\beta(r) = 1/(1 + r^2)$. So $\alpha(0) = 0$; $\beta(0) = 1$ and $(d\alpha/d\mu)(0) = 1$.

Using local Hopf bifurcation theorem, we have the existence of a family of periodic solutions parametrized by r, bifurcating from $r = 0$. Moreover, the lemma 1.1 gives us the second part of the proposition. ■

We denote by: $y(r) = (y_1(r), 0)$ the initial data of the bifurcating periodic solutions of the ODS (2.4) and from proposition 2.2 we see that $\|y(r)\| \leq C\sqrt{r}$; $x(\phi)$ the solution of (0.1) with ϕ as an initial data; x^* the solution of (2.4) such that $x^*(0) = \phi(0)$; T^* the first return time of x^* such that $x_1(T^*) > 0$ and $x_2(T^*) = 0$;

$$\mathcal{B}(y(r)) = \{\phi \in \mathcal{C}([-r, 0], \mathbb{R}^2) \,/\, \|\phi(s) - y(r)\| \leq Cr^{3/2}\}.$$

LEMMA 2.1. There exists a positive real number $r_0 = r_0(C, T)$ such that

$$\|x(\phi)(t)\| \leq Cr^{1/2}$$

for any $r < r_0$ and $\phi \in \mathcal{B}(y(r))$.

Proof. Here, we denote by $x(t)$ the solution $x(\phi)(t)$ for some $\phi \in \mathcal{B}(y(r))$. The RDS (0.1) can be written in the form:

$$\frac{d}{dt}x(t) = Df(0) \cdot x(t - r) + L(x(t - r)) \tag{2.5}$$

where

$$\|L(x)\| \leq M \cdot \|x\|^2 \qquad \text{for } \|x\| \leq 1.$$

We also assume that M is chosen so that

$$\|Df(x)\| \leq M \qquad \text{for } \|x\| \leq 1.$$

Using the inner product in $\mathbb{R}^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \langle x(t), Df(0) \cdot x(t - r)\rangle + \langle x(t), L(x(t - r))\rangle$$

$$= \langle x(t), Df(0) \cdot (x(t - r) - x(t))\rangle + \langle x(t), L(x(t - r))\rangle.$$

Thus

$$\|x(t)\| \leq \|x(0)\| + \int_0^t \|x(s - r) - x(s)\| \, ds + \int_0^t \|L(x(s - r))\| \, ds. \tag{2.6}$$

Starting from a point in $\mathcal{B}(y(r))$ we would like to prove that the solution will never exceed the order of $\sqrt{r}$. We will proceed by contradiction.

Because ϕ is in $\mathcal{B}(y(r))$ we have:

$$\|\phi\| \leq C\sqrt{r} \qquad \text{for some constant } C.$$

Assuming that a solution may become large, it implies that at some point it exceeds the value $2C\sqrt{r}$.

Denote by $\bar{t}$, $r \leq \bar{t} \leq T$ the first time at which it takes this value:

$$\|x(\bar{t})\| = 2C\sqrt{r} \qquad \text{and} \qquad \|X(t)\| \leq 2C\sqrt{r} \qquad \text{for } t \leq \bar{t}.$$

Using inequality (2.6), we get:

$$\|x(\bar{t})\| \leq \|x(0)\| + \int_0^r \|x(s - r) - x(s)\| \, ds + \int_r^{\bar{t}} \|x(s - r) - x(s)\| \, ds + \int_0^{\bar{t}} \|L(x(s - r))\| \, ds.$$

We can assume that r is small enough for $2C\sqrt{r} \leq 1$. We have the following estimates:

$$\|x(0)\| \leq C\sqrt{r}$$

$$\int_0^r \|x(s-r) - x(s)\| \, ds \leq 2Cr^{3/2}$$

$$\int_r^{\bar{t}} \|x(s-r) - x(s)\| \, ds \leq \int_r^T \sup_{\|x\| \leq 1} \|Df(x)\| \cdot r \, ds \leq M \cdot T \cdot r.$$

Finally:

$$\int_0^{\bar{t}} \|L(x(s-r))\| \, ds \leq M \int_0^{\bar{t}} \|x(s-r)\|^2 \, ds \leq 4MC^2Tr.$$

So

$$2C\sqrt{r} \leq C\sqrt{r} + 2Cr^{3/2} + (1 + 4C^2)M \cdot T \cdot r,$$

C being independent of r, we see that for r small enough this inequality cannot be satisfied.

Precisely, we can find a number $r_0 > 0$, $r_0 = r_0(C, T)$, such that: for $r < r_0$ the inequality is not satisfied.

Therefore, we get: $\|x(t)\| \leq 2C\sqrt{r}$ for $-r \leq t \leq T$, and $r \leq r_0$. ■

Proof of proposition 2.1. Note that for any $t \in [3r, T]$, the solution $x(t)$ of (0.1) is two times continuously differentiable.

Writing the Taylor development of $x(t-r)$ and $f(x(t-r))$ in the neighborhood of t and $x(t)$ respectively for $t \in [3r, T]$, we obtain:

$$x(t-r) \simeq x(t) - r\frac{d}{dt}x(t)$$

$$f(x(t-r)) \simeq f(x(t)) - rDf(x(t)) \cdot \frac{d}{dt}x(t).$$

Then:

$$\frac{d}{dt}x(t) \simeq [I + rDf(x(t))]^{-1} \cdot f(x(t)).$$

Note that $f(x(t-r))$ may be written as:

$$f(x(t-r)) = g(r, x(t)) + H(x_t)$$

$g(r, x)$ is defined by (2.2) and $H(x_t)$ is given by (2.3).

Since $x(t)$ is a two times continuously differentiable function, we can develop $x(t-r)$:

$$x(t-r) = x(t) - r\frac{d}{dt}x(t) + \frac{r^2}{2}\frac{d^2}{dt^2}x(t) + o(r^2)$$

$o(r^2)$ is small uniformly in x when $\sup_{-r \leq t \leq T} \|x(t)\| < M$, for each $M \geq 0$. Also we have:

$$f(x(t-r)) = f(x(t)) - rDf(x(t))f(x(t-r)) + r^2[Df(x)^2 \cdot \frac{d}{dt}x(t)$$
$$+ \frac{r^2}{2}D^2f(x(t)) \cdot \left[\frac{d}{dt}x(t)\right]^2 + o(r^2).$$

This implies that:

$$f(x(t-r)) = [I + rDf(x(t))]^{-1} \cdot \left[f(x(t)) + \frac{r^2}{2}(Df(x(t)))^2 \frac{\mathrm{d}}{\mathrm{d}t}x(t) + \frac{r^2}{2} D^2 f(x(t)) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right]^2 + o(r^2) \right].$$

Then:

$$f(x(t-r)) = g(r, x(t)) + [I + rDf(x(t))]^{-1} \left[r(Df(x(t)))^2 \frac{\mathrm{d}}{\mathrm{d}t}x(t) + \frac{r^2}{2} D^2 f(x(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right)^2 + o(r^2) \right].$$

Substituting the above expression for $f(x(t-r))$ in (2.3) for $\phi - x_t$ we obtain:

$$H(x_t) = [I + rDf(x(t))]^{-1} \left[r(Df(x(t)))^2 \frac{\mathrm{d}}{\mathrm{d}t}x(t) + \frac{r^2}{2} Df(x(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right)^2 + o(r^2) \right].$$

For r small enough we have:

$$[I + rDf(x)]^{-1} = I - rDf(x) + r^2(Df(x))^2 + \cdots$$

$$H(x_t) = r^2(Df(x(t)))^2 \frac{\mathrm{d}}{\mathrm{d}t}x(t) + \frac{r^2}{2} D^2 f(x(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right)^2 - \frac{r^3}{2}(Df(x(t))) \frac{3\mathrm{d}}{\mathrm{d}t}x(t) - \frac{r^3}{2} Df(x(t) D^2 f(x(t)) \frac{\mathrm{d}}{\mathrm{d}t}x(t) + o(r^2)$$

if $\sup_{-r \le t \le 0} \|x(t)\| < M$ then $H(x_t) = o(r^2)$, where $o(r^2)$ is small uniformly in x.

Now, let t be in the interval $[0, 3r]$ and $x(t) = x(\phi)(t)$ for some $\phi \in \mathcal{B}(y(r))$. We have:

$$H(x_t) = f(x(t-r)) - g(r, x(t))$$

where $g(r, x)$ is defined by formula (2.2).

But we can write $g(r, x(t))$ in the form:

$$g(r, x(t)) = f(x(t)) + O(r^{3/2}).$$

Then

$$H(x_t) = f(x(t-r)) - f(x(t)) + O(r^{3/2})$$

$$\|H(x_t)\| \le \|f(x(t-r)) - f(x(t))\| + O(r^{3/2}).$$

Since f is a smooth function, we deduce that:

$$\|H(x_t)\| \le M \|x(t-r) - x(t)\| + O(r^{3/2}).$$

Using lemma 2.1 we obtain: $\|H(x_t)\| \le C(T) r^{3/2}$ where $C(T)$ is a positive constant independent of r. ■

We will give some comparison results between x^* and $x(\phi)$.

LEMMA 2.2. For any $\phi \in \mathcal{B}(y(r))$ we have

$$\|x^*(t) - x(\phi)(t)\| = o(r^2).$$

Proof. The RDS (0.1) can be written as:

$$\frac{d}{dt} x(t) = g(r, x(t)) + H(x_t).$$

Then we have:

$$\frac{d}{dt}[x(\phi) - x^*](t) = g(r, x(\phi)(t)) - g(r, x^*(t)) + H(x_t(\phi)).$$

Using the inner product in $\mathbb{R}^2$, we get:

$$\frac{1}{2}\frac{d}{dt}\|x(\phi) - x^*\|^2 \leq M\|x(\phi) - x^*\| + H(x_t(\phi))\|x(\phi) - x^*\|,$$

from which it follows that

$$D^+\|x(\phi) - x^*\| \leq M\|x(\phi) - x^*\| + \|H(x_t(\phi))\|.$$

Here D^+ denotes the derivative from the right.

Using the Gronwall lemma and in view of $x(\phi)(0) = x^*(0)$, we obtain:

$$\|x(\phi)(t) - x^*\|(t) \leq \int_0^t e^{M(t-s)}\|H(x_s(\phi))\|\, ds.$$

So

$$\int_0^t e^{M(t-s)}\|H(x_s(\phi))\|\, ds \leq \int_0^r e^{M(t-s)}\|H(x_s(\phi))\|\, ds + \int_r^t e^{M(t-s)}\|H(x_s(\phi))\|\, ds.$$

From proposition 2.1, we have:

$$\int_0^r e^{M(t-s)}\|H(x_s(\phi))\|\, ds \leq \int_0^r e^{M(t-s)} O(r^{3/2})\, ds \leq Cr^{5/2}$$

and

$$\int_r^t e^{M(t-s)}\|H(x_s(\phi))\|\, ds \leq \int_r^t e^{M(t-s)} o(r^2)\, ds = o(r^2).$$

Thus:

$$\|x(\phi) - x^*\| = o(r^2). \quad \blacksquare$$

LEMMA 2.3. There are two positive real numbers b (independent of r) and $T^\#$ such that:

$$|T^* - T^\#| < b \cdot r^{3/2} \qquad \text{and} \qquad x_2(\phi)(T^\#) = 0.$$

Proof. We construct $T^\#$ in an interval $[T_1, T_2]$, near T^*, T_1 and T_2 are two real numbers such that $x_2(\phi)(T_1)$ [respectively $x_2(\phi)(T_2)$] is positive [respectively negative].

Since $\|x(\phi)(t) - x^*(t)\| = o(r^2)$, uniformly in any bounded set in t, we look for T_1 and T_2 such that $x^*(T_1) \geq Cr^2$ and $x(T_2) \leq -Cr^2$. The velocity of rotation of x^* around 0 is determined by the linear part of the equation:

$$\frac{d}{dt}x(t) = D_x g(r, 0) \cdot x(t) + o(x(t)) \tag{2.7}$$

the solution $x(t)$ of the linear equation associated to (2.7) is given by:

$$x(t) = e^{\alpha(r)t}\begin{bmatrix} \cos\beta(r)t & \sin\beta(r)t \\ -\sin\beta(r)t & \cos\beta(r)t \end{bmatrix} \cdot x(0). \tag{2.8}$$

If we transform (2.8) into polar coordinate, the solution will be:

$$\begin{cases} \rho(t) = e^{\alpha(r)t}\rho(0) \\ \theta(t) = \beta(r)t. \end{cases}$$

We see that:

$$\theta(T^*) \simeq 2\pi$$

set $T_1 = T^* - \varepsilon$ and $T_2 = T^* + \varepsilon$ where ε is a small positive real number, we have:

$$\theta(T_2) \geq 2\pi + \frac{\varepsilon}{2}; \qquad \theta(T_1) \leq 2\pi - \frac{\varepsilon}{2}$$

and

$$\rho(T_1) \geq C_1\sqrt{r}; \qquad \rho(T_2) \geq C_2\sqrt{r},$$

where C_1 and C_2 are positive constant.

For $\varepsilon > 0$ sufficiently small, we obtain:

$$x_2^*(T_1) \geq C\sqrt{r}\frac{\varepsilon}{4} \qquad \text{and} \qquad x_2^*(T_2) \geq C\sqrt{r}\frac{\varepsilon}{4}.$$

For the retarded system, we have:

$$x_2(\phi)(T_2) \geq C\sqrt{r}\frac{\varepsilon}{4} - K_0 r^2 \qquad \text{and} \qquad x_2(\phi)(T_1) \leq C_0\sqrt{r}\frac{\varepsilon}{4} + K_0 r^2$$

where K_0 is a positive constant which is independent of r. We can choose ε such that:

$$\varepsilon > \frac{4K_0 r^2}{C_1\sqrt{r}} = br^{3/2}.$$

Thus:

$$|T^\# - T^*| < br^{3/2}. \quad \blacksquare$$

LEMMA 2.4. There exists a real constant $a^0 > 0$ which is independent of r such that

$$\|x^*(T^*) - x^*(T^\#)\| < a^0 r^2.$$

Proof.

$$\|x^*(T^*) - x^*(T^\#)\| \leq \sup_{T^\prime - r \leq t \leq T^\prime} \left\|\frac{d}{dt}x^*(t)\right\| \cdot |T^* - T^\#|.$$

From proposition 2.2, it follows that the amplitude of x^* is of the order of $\sqrt{r}$. Then there exists a real constant a^0 such that $\sup_{T^\prime - r \leq t \leq T^\prime} \|(d/dt)x^*(t)\| \leq a^0\sqrt{r}$.

Thus we have

$$\|x^*(T^*) - x^*(T^\#)\| \le a^0 r^2. \quad \blacksquare$$

LEMMA 2.5. For any $t \in [T^\# - r, T^\#]$ and any ϕ in $\mathcal{B}(y(r))$,

$$\|x(\phi)(t) - x(\phi)(T^\#)\| \le Cr^{3/2}$$

where C is a positive constant independent of r.

Proof. In the interval $[T^* - r, T^*]$ the solution $x(\phi)$ is continuously differentiable, so that:

$$\|x(\phi)(t) - x(\phi)(T^*)\| \le r \sup_{T^*-r\le t\le T^*} \left\|\frac{\mathrm{d}}{\mathrm{d}t} x(\phi)(t)\right\|.$$

Since: $\|x^*(t) - x(\phi)(t)\| = o(r^2)$ [see lemma 2.2] and $\|x^*(t)\| \le C \cdot \sqrt{r}$, for some constant $C \ge 0$, we have

$$\sup_{T^*-r\le t\le T^*} \left\|\frac{\mathrm{d}}{\mathrm{d}t} x(\phi)(t)\right\| \le C\sqrt{r}.$$

Thus:

$$\|x(\phi)(t) - x(\phi)(T^\#)\| \le Cr^{3/2}. \quad \blacksquare$$

PROPOSITION 2.3. For any $\phi \in \mathcal{B}(y(r))$, $z(\phi) \in \mathcal{B}(y(r))$, where $z(\phi)$ is the restriction of $x(\phi)$ to the interval $[T^\# - r, T^\#]$.

Proof. We first show that $\|x(\phi)(T^\#) - y(r)\| \le Cr^{3/2}$.
In fact we have:

$$\|y(r) - x(\phi)(T^\#)\| \le \|y(r) - x^*(T^*)\| + \|x^*(T^*) - x^*(T^\#)\| + \|x^*(T^\#) - x(\phi)(T^\#)\|.$$

From [1, theorem 2.1], we can see that:

$$\|y(r) - x^*(T^*)\| \le \|y(r) - \phi(0)\| - K_1 r^2, \qquad \text{for some } K_1 \in \mathbb{R}_+^*.$$

On the other hand, we obtain $\|x^*(T^*) - x^*(T^\#)\|$ and $\|x^*(T^\#) - x(\phi)(T^\#)\|$ by lemmas 2.2 and 2.4. Consequently we have:

$$\|y(r) - x(\phi)(T^\#)\| \le \|y(r) - \phi(0)\| - K_1 r^2 + Cr^2 + o(r^2).$$

This implies that $\|y(r) - x(\phi)(T^\#)\| \le Cr^{3/2}$. We will see now that, for any $t \in [T^\# - r, T^\#]$, $\|x(\phi)(t) - y(r)\| \le Cr^{3/2}$. So, let t be an element of $[T^\# - r, T^\#]$, we observe that:

$$\|y(r) - x(\phi)(t)\| \le \|y(r) - x(\phi)(T^\#)\| + \|x(\phi)(T^\#) - x(\phi)(t)\|.$$

Now, in view of lemma 2.5, we have:

$$\|x(\phi)(T^\#) - x(\phi)(t)\| \le Cr^{3/2}.$$

Then, from the above estimate of $\|y(r) - x(\phi)(T^\#)\|$ we deduce that:

$$\|x(\phi)(t) - y(r)\| \le Cr^{3/2} \qquad \text{for any } t \in [T^\# - r, T^\#].$$

This shows that $z(\phi) \in \mathcal{B}(y(r))$. $\blacksquare$

THEOREM 2.1. Under the assumptions (H_1) and (H_2), the RDS (0.1) has at least one periodic solution.

Proof. The proof of the theorem follows from the above proposition and lemmas 2.2, 2.3 and 2.4.

In fact we define the Poincaré operator:

$$\mathcal{P}: \mathcal{B}(y(r)) \to \mathcal{C}([-r, 0], \mathbb{R}^2)$$
$$\phi \to z(\phi)$$

where $z(\phi)$ is the restriction of $x(\phi)$ to the interval $[T^{\#} - r, T^{\#}]$. Proposition 2.3 shows that $\mathcal{P}$ is defined from $\mathcal{B}(y(r))$, (which is a convex bounded set) into itself and that $\mathcal{P}$ is continuous and compact. So using the second Schauder fixed point theorem we conclude that $\mathcal{P}$ has at least one fixed point which corresponds to a periodic solution of the RDS (0.1). ■

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