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On stability of a class of neutral type functional differential equations

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Abstract

This paper deals with the study of stability and estimation of stability domain for a class of nonlinear integro-difference equations, which coincide with special class of neutral type functional differential equations. The new approach for stability study called the pattern equation method is proposed. © 1998 IMACS/Elsevier Science B.V.

Keywords: Stability; Neutral type equations; Integro-difference equations

1. Introduction

The aim of this paper is to investigate stability of certain types of dynamical systems, described by integral or integro-difference equations. Such equations can be often interpreted as retarded or neutral type functional differential equations [10,11]. But the possibility to write them as an integro-difference equation provides new approaches to study them. We shall denote this class of equations by NIDE.

Certain types of such equations are often used in modelling of biological systems. Cooke et al. in papers [4–6] studied the following model for population dynamics or epidemics. Let $N(t)$ denote the number of individuals in an isolated population at time t. The life-span of every individual is assumed to be a fixed constant L. Assume that the number of birth per unit time is some function of $N(t)$, say $f(N(t))$, or more generally $f(t, N(t))$. Under these assumptions, the growth of the population is governed by the equation

$$N(t) = \int_{t-L}^{t} f(s, N(s))\,\mathrm{d}s \tag{1}$$

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In compartmental analysis of biological systems Gyori [7,8] and Arino et al. [1,2] used the equation

$$x(t) - cx(t-h) = \int_{t-h}^{t} f(s, x(s))\,\mathrm{d}s \tag{2}$$

as a model of fluid stream (for example, blood) throughout different organs assimilated to some boxes connected by pipes. The quantities of fluid entering or running out the boxes depend in nonlinear manner on quantities of fluids inside boxes. For all these models, it is natural to suppose that the function $f(t, x(t))$ is nondecreasing with respect to the second argument and that also $f(t, x(t))=0$. For population dynamics model (1), these assumptions mean that, firstly, the greater is the population, the greater is the number of births per unit time and, secondly, if there is no individual in the population, there is no birth also.

The stability of such models was studied by the Lyapunov direct methods in [6,10,11]. Arino et al. [1,2] used Lyapunov functions based on an order relation for stability study of such equations. Existence of positive periodic, almost periodic or pseudo almost periodic solutions of Eqs. (1) and (2) have been established under different assumptions in [1,4,5,9,12,14].

We shall study stability and asymptotic properties of NIDEs using a new approach – *the pattern equation method*. This method is based on the construction of special scalar equations (called pattern equations) whose solutions are upper bounds of the solutions of the initial many-dimensional NIDE. It permits to give explicit conditions on the coefficients of the initial NIDE which imply required stability properties or desired asymptotic behaviour. For discrete Volterra equations, the pattern equation method is developed in [3,13]

2. Problem statement

Let $\mathbb{R}^n$ be the n-dimensional real vector space with a norm $|.|$ and $C=C([-h, 0], \mathbb{R}^n)$ the space of all continuous functions mapping $[-h, 0]$ onto $\mathbb{R}^n$. The norm of a function $\varphi(\theta)\in C$ is $\|\varphi\| = \sup\{|\varphi(\theta)|, -h \le \theta \le 0\}$.

For an $(n\times n)$-matrix $A(t)$, we denote by $|A(t)|$ the operator norm of this matrix corresponding to the norm $|.|$.

Let us consider a nonlinear neutral type integro-difference equation (NIDE) of the form

$$y(t) = \sum_{i=1}^{k} A_i(t) y(t-h_i) + \int_{t-1}^{t} f(s, y(s))\,\mathrm{d}s, \quad y(t) \in \mathbb{R}^n,\ t \ge 0, \quad y(\theta) = \varphi(\theta),$$
$$-h \le \theta \le 0, \quad h = \max\{h_1, \ldots, h_k, 1\}, \quad h_i > 0 \tag{3}$$

Suppose that the matrices $A_i(t)$ are continuous $(n\times n)$-matrices, $f(s, y(s))$ is continuous in both arguments and verifies a Lipschitz condition in the second argument, $\varphi(\theta)\in C$.

Eq. (3) has a continuous solution $y(t)$ if and only if

$$y(0) = \varphi(0) = \sum_{i=1}^{k} A_i(0)\varphi(-h_i) + \int_{-1}^{0} f(s, \varphi(s))\,\mathrm{d}s \tag{4}$$

We shall suppose that the condition (4) will always be satisfied and we shall denote by C_f the subspace of all functions $\varphi(\theta)\in C$ which verify condition (4).

The solution of problem (3) is also the solution of the following neutral type equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t)-\sum_{i=1}^{k}A_i(t)y(t-h_i)\right]=f(t,y(t))-f(t-1,y(t-1))$$
$$t\geq 0,\quad y(\theta)=\varphi(\theta),-h\leq\theta\leq 0 \tag{5}$$

Conversely, if the initial function $y(\theta)=\varphi(\theta)$ verifies condition (4), then the solution of problem (5) is also the solution of problem (3). Under the assumptions made above, problem (5) and consequently problem (3) has one continuous solution $y(t, \varphi)$ for all continuous initial functions $\varphi(\theta)$ [10,11].

Eq. (3) admits $y(t)=0$ as a trivial solution, if

$$f(t,0)=0. \tag{6}$$

The stability of this solution will be studied under two hypotheses:

H1: any solution $y(t, \varphi)$ of Eq. (3) admits, in a ball $|y(t)|\leq a$, $a\geq 0$, the estimate

$$|y(t)|\leq\sum_{i=1}^{k}|A_i(t)||y(t-h_i)|+\int_{t-1}^{t}K(t,s)\beta(|y(s)|)\,\mathrm{d}s,\quad t\geq 0 \tag{7}$$

with a continuous kernel $K(t, s)$,

H2: the continuous scalar function $\beta(u)$, $\beta:\mathbb{R}^1_+\to\mathbb{R}^1_+$ is a monotone nondecreasing function such that $\beta(0)=0$, $\beta(u)>0$, $u>0$.

3. Pattern equation method

Along with NIDE (3), (7) we shall consider a scalar NIDE of the same form:

$$x(t)=\sum_{i=1}^{k}a_i(t)x(t-h_i)+\int_{t-1}^{t}k(t,s)\alpha(|x(s)|)\,\mathrm{d}s,\quad t\geq 0,\quad x(\theta)=\psi(\theta),$$
$$-h\leq\theta\leq 0,\quad x(t)\in\mathbb{R}^1 \tag{8}$$

Here $a_i(t)>0$, $i=1,\ldots,k$ are continuous functions, $\alpha(u)$ is a continuous monotone nondecreasing function such that $\alpha(0)=0$, $\alpha(u)>0$, $u>0$, the kernel $k(t, s)$ is continuous and positive, the initial function $\psi(\theta)$ is also continuous and positive and verifies condition (4). Under these assumptions, the solution $x(t)$ of problem (8) is positive, $x(t)>0$, $t\geq 0$.

The pattern equation method is based on the following two theorems – the comparison theorem and the theorem on existence of a pattern equation.

Theorem 1 *Suppose that*

$$|A_i(t)|\leq a_i(t),\quad a_i(t)>0,\quad t\geq 0$$

$$\begin{aligned}&K(t,s)\le k(t,s),\quad t\ge 0,\quad t-1\le s\le t\\&\beta(u)\le\alpha(u),\quad u\ge 0,\quad |\varphi(\theta)|<\psi(\theta),\quad -h\le\theta\le 0\end{aligned}\tag{9}$$

then for all $t\geq 0$ *it holds* $|y(t)|\le x(t)$

Proof. At time $t=0$, we have $x(0)=\psi(0)>|y(0)|$. So, there exists a neighbourhood of the point $t=0$ where $x(t)>|y(t)|$. Suppose now that t_1 is the first moment such that $x(t_1)=|y(t_1)|$ and $x(t)>|y(t)|$ for all $t<t_1$. These assumptions lead to the following contradiction

$$\begin{aligned}0&=x(t_1)-|y(t_1)|\ge\sum_{i=1}^{k}a_i(t_1)x(t_1-h_i)+\int_{t_1-1}^{t_1}k(t_1,s)\alpha(x(s))\,\mathrm{d}s-\sum_{i=1}^{k}|A_i(t_1)||y(t-h_i)|\\&\quad-\int_{t_1-1}^{t_1}K(t_1,s)\beta(|y(s)|)\,\mathrm{d}s=\sum_{i=1}^{k}a_i(t_1)(x(t_1-h_i)-|y(t_1-h_i)|)\\&\quad+\sum_{i=1}^{k}(a_i(t_1)-|A_i(t_1)|)|y(t_1-h_i)|+\int_{t_1-1}^{t_1}k(t_1,s)(\alpha(x(s))-\beta(|y(s)|))\,\mathrm{d}s+\int_{t_1-1}^{t_1}(k(t_1,s)\\&\quad-K(t_1,s))\beta(|y(s)|)\,\mathrm{d}s>0\end{aligned}$$

Here the first sum is strictly positive because of $a_i(t)>0$, $x(t_1-h_i)-|y(t_1-h_i)|>0$, $h_i>0$ and all other terms are nonnegative.

Theorem 2 *For any scalar continuous positive function*, $p(t)>0$, $0\le t$, $p(t)=1$, $-h\le t\le 0$, *and satisfying condition* (4) *there exists an infinity of homogeneous scalar NIDE's* (8) *with* $p(t)$ *as a solution*

$$x(t)=p(t),\quad t\ge 0\tag{10}$$

Proof. To prove this theorem, we construct such scalar NIDE's in explicit form. Let us introduce positive numbers γ_i such that

$$\gamma_i>0,\quad i=1,\dots,k+1,\quad \sum_{i=1}^{k+1}\gamma_i=1\tag{11}$$

Consider now, for a given nonlinearity $\alpha(u)$, a homogeneous scalar NIDE of the form

$$x(t)=\sum_{i=1}^{k}\frac{p(t)\gamma_i}{p(t-h_i)}x(t-h_i)+\gamma_{k+1}\int_{t-1}^{t}\frac{p(t)}{\alpha(p(s))}\alpha(|x(s)|)\,\mathrm{d}s\tag{12}$$

By direct substitution we verify that the function (10) is a solution of the NIDE (12) and moreover the coefficients of NIDE (12) satisfy conditions

$$\begin{aligned}&a_i(t)=\tfrac{p(t)}{p(t-h_i)}\gamma_i>0,\quad t\ge 0\\&k(t,\,s)=\tfrac{p(t)\gamma_{k+1}}{\alpha(p(s))}>0,\quad t\ge 0,\ t-1\le s\le t\end{aligned}\tag{13}$$

Condition (4) is also satisfied because of continuity of the function $p(t)$ at the point t=0.

The scalar NIDE (12) will be called a *pattern equation* and the function $p(t)$ – *a pattern function*. Now we can describe the steps of the pattern equation method:

(a) choose a pattern function (i.e. a positive scalar function $p(t)$, $t\geq -h$) in such a way that the solution $x(t)=p(t)$ has the required stability properties or asymptotic behaviour,
(b) using Theorem 2, construct a pattern equation for the chosen function $p(t)$,
(c) compare the coefficients $A_i(t)$, $K(t, s)$, $\beta(u)$ of initial NIDE (3), (7) with the coefficients of the so constructed pattern scalar NIDE (12). If

$$|A_i(t)| \leq \frac{p(t)}{p(t-h_i)}\gamma_i, \quad t \geq 0$$

$$K(t,s) \leq \frac{p(t)\gamma_{k+1}}{\alpha(p(s))}, \quad t \geq 0, \quad t-1 \leq s \leq t$$
$$\beta(u) \leq \alpha(u), \quad u \geq 0, \quad |y(\theta)| < p(\theta), \quad -h \leq \theta \leq 0$$

then according to Theorem 1

$$|y(t)| \leq p(t)$$

So, the solution of initial NIDE (3), (7) will have the required properties described by the chosen function $p(t)$.

4. Stability

Here we study stability properties of NIDE (3), (7). NIDE (3) under the assumption (6) has a trivial solution $y(t)$=0, $t\geq 0$. We consider the usual definition of stability [10,11].

Definition 3 *The trivial solution $y(t)$=0, $t\geq 0$ of NIDE (3) is called Lyapunov stable if for any ε>0 there exists a $\delta(\varepsilon)$>0 such that $|y(t)| \leq \varepsilon$, $t\geq 0$ if $\|\varphi\| \leq \delta$, $\varphi \in C_f$.*

The following theorem gives sufficient conditions for stability in terms of corresponding pattern equations and coefficients of the initial NIDE.

Theorem 4 *Suppose there exist*

1. *a bounded pattern function $p(t)$, $0<p(t)\leq P$, $t\geq -h$,*
2. *numbers γ_i satisfying conditions (11) and a number δ_0>0, $\delta_0 P\leq a$, such that for all $0<\delta\leq\delta_0$, it holds simultaneously*

$$\begin{array}{ll} |A_i(t)| \leq \frac{p(t)}{p(t-h_i)}\gamma_i, & t \geq 0, \\ K(t,s) \leq \frac{\delta p(t)\gamma_{k+1}}{\alpha(\delta p(s))}, & t \geq 0,\ t-1 \leq s \leq t,\ \delta \leq \delta_0, \\ \beta(u) \leq \alpha(u), & u \geq 0, \\ |y(\theta)| < \delta p(\theta), & -h \leq \theta \leq 0. \end{array} \tag{14}$$

Then, the trivial solution of NIDE (3), (7) *is Lyapunov stable.*

Proof. To prove this theorem, we construct the pattern Eq. (10) with a nonlinear function equal to $\beta(u)$ and the pattern function $\delta p(t)$. The solution of this pattern Eq. (13) is $x(t)=\delta p(t)\leq P^{-1}a$. Then, from condition (12) and Theorem 1 it follows, the estimate

$$|y(t)| \leq x(t) = \delta p(t) \leq P\delta. \tag{15}$$

The stability of the trivial solution of NIDE (3), (7) is a direct consequence of the estimate (15).

Formally, the Theorem 4 on stability in the pattern equation method seems to be very close to the Theorem on stability in the Lyapunov direct method [10,11]. In both methods, stability depends on existence and properties of some auxiliary functions. But, in Theorem 4 we can take any bounded function as a pattern function and obtain directly some stability conditions. If the coefficients of estimate (7) for the studied equation satisfy conditions (14) for such a choice of a pattern function, then the trivial solution of NIDE (3), (7) is stable. Otherwise, we can take another bounded pattern function and verify Eq. (14) one more time and so on. This process is very simple, almost mechanical. Contrary to this, the choice of an appropriate Lyapunov function is not so simple and sometimes may be compared to an invention. Evidently, we cannot be sure to establish stability by the pattern equation method for a given Eq. (3) by the described procedure. But the more bounded functions $p(t)$ are taken, the larger the set of stable equations will be.

As a corollary of Theorem 4, if we put $p(t)=1$, and we take γ_i equal to those given in conditions (16), we can obtain the following well-known result [11]

Corollary 5 *The trivial solution of NIDE* (3) *is stable, if*

$$\begin{array}{ll} |A_i(t)| \leq \gamma_i, & t \geq 0,\ i = 1, \ldots, k, \\ \sum_{i=1}^{k} \gamma_i = \gamma < 1, & \\ f(t, y(t)) \leq \beta(|y(t)|) \leq (1-\gamma)|y(t)|, & t \geq 0, \\ |y(t)| \leq a, & \end{array} \tag{16}$$

Conditions (16) are only sufficient for stability but not necessary. Taking another bounded function $p(t)$ we can obtain stability conditions less restrictive than Eq. (16). The set of equations which do not satisfy estimate (16) contains both stable and unstable equations.

Example 1 Let us consider a linear NIDE of the form

$$y(t) = C(t)y(t-\pi) + \int_{t-\pi}^{t} K(t,s)y(s)\,\mathrm{d}s, \quad y(t) \in \mathbb{R}^n, \quad t \geq 0 \tag{17}$$

For Eq. (17), estimate (7) has the form

$$|y(t)| \leqslant |C(t)||y(t-\pi)| + \int_{t-\pi}^{t} |K(t,s)||y(s)|\,\mathrm{d}s. \tag{18}$$

Let us now take the pattern function $p(t)=(2+\sin t)$, $1\leq p(t)\leq 3$, and let us construct the linear pattern equation of the form (18) with $\gamma_1=\gamma_2=0.5$

$$x(t)=\frac{p(t)\gamma_1}{p(t-\pi)}x(t-\pi)+\gamma_2\int_{t-\pi}^{t}\frac{p(t)}{p(s)}x(s)\,\mathrm{d}s=\frac{1}{2}\left(\frac{2+\sin t}{2-\sin t}\right)x(t-\pi)+\frac{1}{2}\int_{t-\pi}^{t}\frac{2+\sin t}{2+\sin s}x(s)\,\mathrm{d}s.$$

If now

$$|C(t)|\leq\frac{1}{2}\left(\frac{2+\sin t}{2-\sin t}\right),\ |K(t,s)|\leq\frac{1}{2}\left(\frac{2+\sin t}{2+\sin s}\right), \tag{19}$$

then, according to Theorem 4, the trivial solution of Eq. (17) is stable. Conditions (19) are quite different from conditions (16). In particular, the upper bound for the norm of matrix $C(t)$ may be equal to 1.5.

5. Asymptotic stability

Definition 6 *The trivial solution* $y(t)=0$ *of NIDE* (3) *is called asymptotically stable if it is stable and*

$$y(t,\varphi)\to 0,\quad t\to\infty,\ \varphi\in Q\subset C_f.$$

The set Q of such initial functions φ is called the attraction domain of the trivial solution. If $Q=C_f$, then the trivial solution is globally asymptotically stable.

Theorem 7 *Suppose there exist*

1. *a bounded pattern function $p(t)$ such that*

$$0<p(t)\leq P,\quad t\geq -h,\quad p(t)\to 0,\quad t\to\infty$$

2. *numbers γ_i, $i=1,\ldots,k+1$, satisfying conditions* (11) *and a number $\delta_0>0$, $\delta_0 P\leq a$ such that for all $0<\delta\leq\delta_0$, conditions* (14) *are satisfied.*

Then, the trivial solution of NIDE (3), (7) *is asymptotically stable and the ball $\|\varphi\|\leq\delta_0$, $\delta_0 P\leq a$, lies in the attraction domain Q. If estimate* (7) *is valid for all $y(t)$, then the trivial solution is globally asymptotically stable.*

Proof. In this case, the pattern Eq. (12) has the form

$$x(t)=\sum_{i=1}^{k}\frac{p(t)\gamma_i}{p(t-h_i)}x(t-h_i)+\int_{t-1}^{t}\frac{\delta p(t)\gamma_{k+1}}{\alpha(\delta p(s))}\alpha(x(s))\,\mathrm{d}s,\quad t\geq 0. \tag{20}$$

The solution of Eq. (20) is equal to $\delta p(t)$, $x(t)=\delta p(t)$. So, if $\|\varphi\|\leq\delta_0$, then, according to comparison Theorem 1, it holds

$$|y(t,\varphi)|\leq x(t)=\delta p(t) \tag{21}$$

The asymptotic stability of the trivial solution of NIDE (3), (7) follows now directly from estimate (21) and from properties of the pattern function $p(t)$.

Example 2 Cooke and Kaplan [6] studied the equation

$$y(t) = \int_{t-h}^{t} a(s)y(s)(1 - y(s))\,\mathrm{d}s \tag{22}$$

with periodic coefficient $a(t)$ and showed that Eq. (22) has a nontrivial periodic solution provided $h\{\inf a(t),\ t\in\mathbb{R}^1\}>1$. Let us now study stability of the trivial solution $y(t)=0$ to the Eq. (22). Suppose the $\{\sup a(t),\ t\in\mathbb{R}^1\}\leq a$, then we have an estimate

$$|a(t)y(t)(1 - y(t))| \leq a|y(t)|(1 + |y(t)|) = K(t,s)\beta(|y(t)|),$$

$$K(t,s) = a, \quad \beta(u) = u(1 + u).$$

Take now as a pattern function $p(t)=1$, $t\leq 0$, $p(t)=(1+t)^{-1}$, $t\geq 0$, and $\gamma_{k+1}=h^{-1}$. If

$$K(t,s) = a \leq h^{-1}(1 + \delta_0)^{-1} = \inf\left\{\frac{p(t)}{\beta(\delta p(s))},\ t \geq 0,\ t - 1 \leq s \leq t,\ \delta \leq \delta_0 < 1\right\}, \tag{23}$$

then we have the estimate $|y(t)|\leq\delta p(t)$, if $\|\varphi\|\leq\delta<\delta_0$, which implies the asymptotic stability. So, the conditions (23) give a relationship between the maximal value of the coefficients $a(t)$, the delay h and the upper bound of the initial function $\varphi(\theta)$ which yields the asymptotic stability.

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