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Convergence in Asymptotically Autonomous Functional Differential Equations

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In this paper, we consider linear and nonlinear perturbations of a linear autonomous functional differential equation which has infinitely many equilibria. We give sufficient conditions under which the solutions of the perturbed equation tend to the equilibria of the unperturbed equation at infinity. As a consequence, we obtain sufficient conditions for systems of delay differential equations to have asymptotic equilibrium. © 1999 Academic Press

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1. INTRODUCTION

The present paper was motivated by the following result of Győri and the second author:

THEOREM A [4, Theorem 2]. *Consider the scalar linear delay differential equation*

$$\dot{x}(t) = (\tilde{c} + c(t))(x(t) - x(t-\tau)) + d(t)x(t-\sigma), \tag{1.1}$$

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where $\tilde{c} \in \mathbb{R}$, $\tau, \sigma \geq 0$ *are constants and* $c, d\colon [0, \infty) \to \mathbb{R}$ *are continuous functions. Suppose that*

$$\tilde{c}\tau < 1, \tag{1.2}$$

$$c(t) \to 0 \quad \text{as } t \to \infty, \tag{1.3}$$

and

$$\int_0^\infty |d(t)|\, dt < \infty. \tag{1.4}$$

Then Eq. (1.1) *has asymptotic equilibrium; i.e., statements* (i) *and* (ii) *below hold.*

(i) *Every solution of* (1.1) *tends to a constant at infinity.*

(ii) *For every* $\xi \in \mathbb{R}$, *Eq.* (1.1) *has a solution* x *such that* $x(t) \to \xi$ *as* $t \to \infty$.

Results of this type can be used to establish asymptotic formulae for the solutions of delay differential equations (see [4]).

In Theorem A, Eq. (1.1) is considered as a perturbation of the autonomous equation

$$\dot{x}(t) = \tilde{c}(x(t) - x(t - \tau)). \tag{1.5}$$

Note that every constant function is a solution of the latter equation. The role of Assumption (1.2) is to guarantee that any solution of (1.5) tends to a constant as $t \to \infty$. Indeed, a simple analysis of the characteristic equation

$$\lambda = \tilde{c}(1 - e^{-\lambda\tau}) \tag{1.6}$$

for Eq. (1.5) shows that Assumption (1.2) is equivalent to the fact that $\lambda_0 = 0$ is a simple root of (1.6) and any other root of (1.6) has negative real part which, by known results from the theory of linear autonomous functional differential equations (see, e.g., [6, 7]), implies that the solutions of (1.5) are *asymptotically constant*.

In this paper, among others, we prove the following generalization of Theorem A to systems of delay differential equations:

THEOREM B. *Consider the system*

$$\dot{x}(t) = \sum_{i=1}^{k} \big(\tilde{C}_i + C_i(t)\big)\big(x(t - \omega_i) - x(t - \tau_i)\big) + \sum_{j=1}^{l} D_j(t)x(t - \sigma_j), \tag{1.7}$$

where ω_i, τ_i, $i = 1, \ldots, k$, *and* σ_j, $j = 1, \ldots, l$ *are nonnegative constants*, $\tilde{C}_i$, $i = 1, \ldots, k$, *are constant* $n \times n$ *matrices*, $C_i(t)$, $0 \le t < \infty$, $i = 1, \ldots, k$, *and* $D_j(t)$, $0 \le t < \infty$, $j = 1, \ldots, l$, *are continuous* $n \times n$ *matrix functions. Suppose that every solution of the autonomous system*

$$\dot{x}(t) = \sum_{i=1}^{k} \tilde{C}_i(x(t - \omega_i) - x(t - \tau_i)) \tag{1.8}$$

is asymptotically constant (*tends to a constant vector at infinity*). *If*

$$|C_i(t)| \to 0 \quad \text{as } t \to \infty,\ i = 1, \ldots, k, \tag{1.9}$$

and

$$\int_0^\infty |D_j(t)|\, dt < \infty, \qquad j = 1, \ldots, l, \tag{1.10}$$

then System (1.7) *has asymptotic equilibrium, i.e.*,

(i) *Every solution of* (1.7) *tends to a constant vector at infinity.*

(ii) *For every* $\xi \in \mathbb{R}^n$, *Eq.* (1.7) *has a solution* x *such that* $x(t) \to \xi$ *as* $t \to \infty$.

Remark. It follows from the results of Atkinson and Haddock (see [2, Theorem 3.1]) that every solution of Eq. (1.8) is asymptotically constant if $r\sum_{i=1}^{k}|\tilde{C}_i| < 1$, where $r = \max\{\omega_1, \ldots, \omega_k, \tau_1, \ldots, \tau_k\}$ and $|\cdot|$ is the matrix norm induced by the norm used in $\mathbb{R}^n$.

In fact, we prove a more general result concerning linear and possibly nonlinear perturbations of linear autonomous functional differential equations having infinitely many equilibria. Our main results, formulated in Section 2, give sufficient conditions under which the solutions of the perturbed system tend to the equilibria of the unperturbed equation at infinity.

We remark that the generalization of Theorem A to systems of delay differential equations is nontrivial which is mainly due to the following two facts:

1. One of the main steps of the proof of Theorem A in [4] is to show that the solutions of the "balanced" equation

$$\dot{x}(t) = (\tilde{c} + c(t))(x(t) - (t - \tau)) \tag{1.11}$$

are uniformly stable and asymptotically constant. The proof presented in [4] (see [4, Lemma 7]) strongly uses the scalar nature of Eq. (1.11) and the fact that in Eq. (1.11) there is only one delay.

2. The proof of statement (ii) of Theorem A in [4] is accomplished by showing that Eq. (1.11) has a solution with a nonzero limit at infinity.

Evidently, for scalar linear equations this is equivalent to statement (ii). However, for systems or nonlinear equations this is not true. In this case, for every constant vector ξ, we have to show the existence of a solution x of the *terminal value problem*

$$x(t) \to \xi \quad \text{as } t \to \infty. \tag{1.12}$$

(For other results on the terminal value problem and asymptotic constancy for functional differential equations, see [1−3, 5, 8, 9] and the references therein.) Therefore, the proof of Theorem B requires different arguments. The proof of the asymptotic constancy and uniform stability of the solutions of the "balanced equation" (cf. Theorem 1 below) is based on the abstract variation-of-constants formula and the decomposition theory of linear autonomous functional differential equations (see [6, Chap. 7]). The solution of the terminal value problem (1.12) is found as a fixed point of an appropriate integral operator which can be obtained from the decomposition in the variation-of-constants formula.

2. MAIN RESULTS

Let $|\cdot|$ denote any norm in $\mathbb{R}^n$. Given $r \geq 0$, let $C = C([-r,0],\mathbb{R}^n)$ be the Banach space of continuous functions from $[-r,0]$ into $\mathbb{R}^n$ with the supremum norm, $\|\phi\| = \sup_{-r \leq \theta \leq 0}|\phi(\theta)|$ for $\phi \in C$.

Consider the linear autonomous functional differential equation

$$\dot{x}(t) = L(x_t), \tag{2.1}$$

where $L\colon C \to \mathbb{R}^n$ is linear and continuous and $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

We deal with perturbations of Eq. (2.1) of the form

$$\dot{y}(t) = L(y_t) + M(t, y_t), \tag{2.2}$$

and

$$\dot{z}(t) = L(z_t) + M(t, z_t) + f(t, z_t). \tag{2.3}$$

In Eqs. (2.2) and (2.3), $M\colon [0,\infty) \times C \to \mathbb{R}^n$ is continuous, for each $t \geq 0$, $M(t,\cdot)\colon C \to \mathbb{R}^n$ is linear and such that

$$|M(t,\phi)| \leq \mu(t)\|\phi\|, \qquad t \geq 0,\ \phi \in C, \tag{2.4}$$

where μ is a nonnegative continuous function on $[0,\infty)$. The nonlinearity $f\colon [0,\infty) \times C \to \mathbb{R}^n$ is assumed to be continuous and Lipschitzian with respect to its second variable, i.e.,

$$\left| f(t, \phi_1) - f(t, \phi_2) \right| \le \gamma(t) \|\phi_1 - \phi_2\|, \qquad t \ge 0,\ \phi_1, \phi_2 \in C, \quad (2.5)$$

where γ is nonnegative and continuous on $[0, \infty)$.

Under the above assumptions, for every $\sigma \ge 0$, $\phi \in C$, Eqs. (2.1)–(2.3) have a unique solution with initial value ϕ at σ, denoted by $x(\sigma, \phi)$, $y(\sigma, \phi)$, and $z(\sigma, \phi)$, respectively, (see [6, Theorem 2.2.3]).

Let

$$E = \left\{ \xi \in \mathbb{R}^n | L(\phi_\xi) = 0 \right\},$$

where ϕ_ξ is the corresponding constant function in C defined by

$$\phi_\xi(\theta) = \xi \quad \text{for } \theta \in [-r, 0].$$

Throughout the paper, we assume the following assumption:

(H) Equation (2.1) has infinitely many equilibria and every solution of (2.1) approaches some equilibrium point as $t \to \infty$.

From the theory of linear autonomous functional differential equations (see [6, Chap. 7; 7, Chap. 7]), it follows that assumption (H) is satisfied if and only if any root of the characteristic equation

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda I - L(e^{\lambda \cdot} I) \qquad (I \text{ is the unit matrix}),$$

different from $\lambda_0 = 0$, has negative real part and the *ascent* of the characteristic root $\lambda_0 = 0$ (the order of λ_0 as a pole of Δ^{-1}) equals one.

Our aim in this paper is to find conditions on M and f under which the solutions of Eqs. (2.2) and (2.3) tend to the equilibria of Eq. (2.1) as $t \to \infty$.

Our main results are formulated in the following two theorems. The first theorem deals with the linear perturbation (2.2). It shows that the above conclusion is true if Eqs. (2.1) and (2.2) have the same equilibria (see Assumption (2.6) below) and μ vanishes at infinity.

THEOREM 1. *Let assumption* (H) *hold. Suppose that*

$$M(t, \phi_\xi) = 0 \quad \textit{for every } \xi \in E, \tag{2.6}$$

and

$$\mu(t) \to 0 \quad \textit{as } t \to \infty. \tag{2.7}$$

Then the following statements are valid.

(i) *Every solution of Eq.* (2.2) *tends to some* $\xi \in E$ *at infinity.*

(ii) *The zero solution of Eq.* (2.2) *is uniformly stable.*

For the nonlinear perturbation (2.3), we prove

THEOREM 2. *In addition to the assumptions of Theorem* 1, *assume that*

$$\int_0^\infty |f(t,0)|\,dt < \infty, \tag{2.8}$$

and

$$\int_0^\infty \gamma(t)\,dt < \infty. \tag{2.9}$$

Then the following statements are valid.

(i) *Every solution of Eq.* (2.3) *tends to some* $\xi \in E$ *at infinity.*

(ii) *For every* $\xi \in E$, *Eq.* (2.3) *has a solution* z *such that* $z(t) \to \xi$ *as* $t \to \infty$.

In the case of System (1.7), the above symbols are listed below,

$$\begin{aligned}
L(\phi) &= \sum_{i=1}^{k} \tilde{C}_i(\phi(-\omega_i) - \phi(-\tau_i)),\\
E &= \mathbb{R}^n,\\
\Delta(\lambda) &= \lambda I - \sum_{i=1}^{k} \tilde{C}_i(e^{-\lambda\omega_i} + e^{-\lambda\tau_i}),\\
M(t,\phi) &= \sum_{i=1}^{k} C_i(t)(\phi(-\omega_i) - \phi(-\tau_i)),\\
\mu(t) &= 2\sum_{i=1}^{k} |C_i(t)|,\\
f(t,\phi) &= \sum_{j=1}^{l} D_j(t)\phi(-\sigma_j),\\
\gamma(t) &= \sum_{j=1}^{l} |D_j(t)|.
\end{aligned}$$

Thus, Theorem B in the Introduction is an immediate consequence of Theorem 2.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Let $\sigma \geq 0$, $\phi \in C$ be arbitrary. By the variation-of-constants formula (cf. [6, Chap. 6]), the solution $y = y(\sigma, \phi)$ of Eq. (2.2) can be written in the form

$$y_t = T(t-\sigma)\phi + \int_\sigma^t T(t-\tau) X_0 M(\tau, y_\tau)\, d\tau, \qquad t \geq \sigma, \tag{3.1}$$

where $T(t)$: $C \to C$ is the solution operator for Eq. (2.1) given by

$$T(t)\phi = x_t(0, \phi), \qquad t \geq 0,\ \phi \in C,$$

and X_0 is the $n \times n$ matrix function defined on $[-r, 0]$ by

$$X_0(\theta) = \begin{cases} 0 & \text{for } -r \leq \theta < 0, \\ I & \text{for } \theta = 0. \end{cases}$$

The state space C can be decomposed into a direct sum, $C = P \oplus Q$, where P is the generalized eigenspace corresponding to the characteristic value $\lambda_0 = 0$ of Eq. (2.1) and Q is the complementary subspace which is invariant under the family of operators $T(t)$, $t \geq 0$ (cf. [6, Chap. 7]). That is, any $\phi \in C$ can be written uniquely as

$$\phi = \phi^P + \phi^Q, \tag{3.2}$$

where $\phi^P \in P$ and $\phi^Q \in Q$ denote the projections of ϕ onto subspaces P and Q, respectively.

By assumption (H), P consists of the equilibria of (2.1), i.e.,

$$P = \{\phi_\xi \,|\, \xi \in E\}. \tag{3.3}$$

Thus, $T(t)$ on P may be defined for all values $t \in (-\infty, \infty)$. In [6, Chap. 7] it is shown that the assumption on the characteristic values of Eq. (2.1) implies that there exist constants $K > 0$, $\alpha > 0$ such that

$$\begin{aligned} \|T(t)\phi^P\| &\leq K\|\phi\|, && -\infty < t < \infty,\ \phi \in C, \\ \|T(t)X_0^P\| &\leq K, && -\infty < t < \infty, \\ \|T(t)\phi^Q\| &\leq Ke^{-\alpha t}\|\phi\|, && t \geq 0,\ \phi \in C, \\ \|T(t)X_0^Q\| &\leq Ke^{-\alpha t}, && t \geq 0. \end{aligned} \tag{3.4}$$

If we make the decomposition (3.2) in the variation-of-constants formula (3.1), we obtain an equivalent system (cf. [6, Theorem 7.6.1]),

$$y_t^P = T(t-\sigma)\phi^P + \int_\sigma^t T(t-\tau)X_0^P M(\tau, y_\tau)\,dt, \tag{3.5a}$$

$$y_t^Q = T(t-\sigma)\phi^Q + \int_\sigma^t T(t-\tau)X_0^Q M(\tau, y_\tau)\,d\tau, \tag{3.5b}$$

$$y_t = y_t^P + y_t^Q, \qquad t \geq \sigma. \tag{3.5c}$$

By virtue of (3.2) and the linearity of $M(\tau, \cdot\,)$ we have

$$M(\tau, y_\tau) = M\big(\tau, y_\tau^P + y_\tau^Q\big) = M\big(\tau, y_\tau^P\big) + M\big(\tau, y_\tau^Q\big).$$

But, in view of (2.6) and (3.3), $M(\tau, y_\tau^P) = 0$. Hence

$$M(\tau, y_\tau) = M\big(\tau, y_\tau^Q\big), \qquad \tau \geq \sigma. \tag{3.6}$$

Consequently, System (3.5) is equivalent to the equations

$$y_t^P = T(t-\sigma)\phi^P + \int_\sigma^t T(t-\tau)X_0^P M\big(\tau, y_\tau^Q\big)\,d\tau, \tag{3.7a}$$

$$y_t^Q = T(t-\sigma)\phi^Q + \int_\sigma^t T(t-\tau)X_0^Q M\big(\tau, y_\tau^Q\big)\,d\tau \tag{3.7b}$$

for $t \geq \sigma$.

From (3.7b), in view of (2.4) and estimates (3.4), we obtain

$$\|y_t^Q\| \leq K\|\phi\|e^{-\alpha(t-\sigma)} + K\sup_{\tau\geq\sigma}\mu(\tau)\int_\sigma^t e^{-\alpha(t-\tau)}\|y_\tau^Q\|\,d\tau, \qquad t \geq \sigma.$$

Choose $\beta \in (0, \alpha)$. Multiplying the latter inequality by $e^{\beta(t-\sigma)}$, we get

$$\begin{aligned}\|y_t^Q\|e^{\beta(t-\sigma)} &\leq K\|\phi\|e^{-(\alpha-\beta)(t-\sigma)}\\ &\quad + K\sup_{\tau\geq\sigma}\mu(\tau)\int_\sigma^t e^{-(\alpha-\beta)(t-\tau)}\|y_\tau^Q\|e^{\beta(\tau-\sigma)}\,d\tau\end{aligned} \tag{3.8}$$

for $t \geq \sigma$. Define

$$v(t) = \sup_{\sigma\leq\tau\leq t}\|y_\tau^Q\|e^{\beta(\tau-\sigma)}, \qquad t \geq \sigma.$$

From (3.8), it follows

$$\|y_t^Q\| e^{\beta(t-\sigma)} \le K\|\phi\| + K \sup_{\tau\ge\sigma} \mu(\tau) v(t) \int_\sigma^t e^{-(\alpha-\beta)(\tau-\sigma)}\, d\tau$$
$$\le K\|\phi\| + K(\alpha-\beta)^{-1} \sup_{\tau\ge\sigma} \mu(\tau) v(t)$$

for $t \ge \sigma$. Taking into account that v is nondecreasing, the latter inequality implies

$$v(t) \le K\|\phi\| + K(\alpha-\beta)^{-1} \sup_{\tau\ge\sigma} \mu(\tau) v(t), \qquad t \ge \sigma. \tag{3.9}$$

If σ_0 is so large that

$$\sup_{\tau\ge\sigma_0} \mu(\tau) < K^{-1}(\alpha-\beta),$$

(in view of (2.7) such a constant certainly exists) and $\sigma \ge \sigma_0$, then (3.9) yields

$$v(t) \le \varkappa_1 \|\phi\|, \qquad t \ge \sigma,$$

where $\varkappa_1 = K[1 - K(\alpha-\beta)^{-1} \sup_{\tau\ge\sigma_0} \mu(\tau)]^{-1}$. Consequently,

$$\|y_t^Q\| \le \varkappa_1 \|\phi\| e^{-\beta(t-\sigma)}, \qquad t \ge \sigma \ge \sigma_0. \tag{3.10}$$

By virtue of (3.3), $T(t)\phi^P$ is independent of t. Therefore, from (3.7a), in view of estimates (3.4) and (3.10), we obtain, for $t_2 \ge t_1 \ge \sigma \ge \sigma_0$,

$$\|y_{t_1}^P - y_{t_2}^P\| \le K \sup_{\tau\ge\sigma_0} \mu(\tau) \varkappa_1 \|\phi\| \int_{t_1}^{t_2} e^{-\beta(\tau-\sigma)}\, d\tau$$
$$\le K \sup_{\tau\ge\sigma_0} \mu(\tau) \varkappa_1 \|\phi\| \beta^{-1} e^{-\beta(t_1-\sigma)}.$$

From this, the Cauchy criterion assures the existence of the limit $\psi = \lim_{t\to\infty} y_t^P$ in C. Since P is a finite-dimensional subspace of C, it is closed in C and hence $\psi \in P$; i.e., $\psi = \phi_\xi$ for some $\xi \in E$. Since $y_t^Q \to 0$ (cf. (3.10)), $y_t \to \phi_\xi$ as $t \to \infty$. This completes the proof of statement (i).

To show statement (ii), observe that from (3.7a), by (3.10) and by similar estimates as before, we obtain

$$\|y_t^P\| \le K\|\phi\| + K \sup_{\tau\ge\sigma} \mu(\tau) \varkappa_1 \|\phi\| \int_\sigma^t e^{-\beta(\tau-\sigma)}\, d\tau, \qquad t \ge \sigma.$$

Hence

$$\|y_t^P\| \le \varkappa_2 \|\phi\|, \qquad t \ge \sigma \ge \sigma_0, \tag{3.11}$$

where $\varkappa_2 = K[1 + \varkappa_1 \beta^{-1} \sup_{\tau \ge \sigma_0} \mu(\tau)]$. Combining (3.5c), (3.10), and (3.11), we conclude

$$\|y_t\| = \| y_t(\sigma, \phi) \| \le \varkappa \|\phi\|, \qquad t \ge \sigma \ge \sigma_0,\ \phi \in C, \tag{3.12}$$

where the constant $\varkappa = \varkappa_1 + \varkappa_2$ is independent of σ and ϕ. Therefore, the zero solution of Eq. (2.2) is uniformly stable on $[\sigma_0, \infty)$. Since the uniform stability on the compact interval $[0, \sigma_0]$ follows by standard estimates on the growth of the solutions of linear functional differential equations (cf. [6, Theorem 6.1.1] and its proof), the zero solution of (2.2) is uniformly stable on the whole interval $[0, \infty)$. The proof of the theorem is complete.

Before we present the proof of Theorem 2, we establish some lemmas regarding L_1-functions.

The function space $L_1(0, \infty)$ consists of the Lebesgue measurable functions $g\colon (0, \infty) \to \mathbb{R}$ such that $\int_0^\infty |g(t)|\, dt < \infty$. With the norm

$$\|g\|_{L_1(0,\infty)} \overset{\text{def}}{=} \int_0^\infty |g(t)|\, dt, \qquad g \in L_1(0, \infty),$$

$L_1(0, \infty)$ is a Banach space.

LEMMA 1. *Let $\alpha > 0$ and $\gamma \in L_1(0, \infty)$. Then the convolution*

$$g(t) \overset{\text{def}}{=} \int_0^t e^{-\alpha(t-\tau)} \gamma(\tau)\, d\tau, \qquad t \ge 0$$

has the following properties,

$$g \text{ is continuous on } [0, \infty), \tag{3.13}$$

$$g(t) \to 0 \quad \text{as } t \to \infty, \tag{3.14}$$

$$g \in L_1(0, \infty) \quad \text{and} \quad \|g\|_{L_1(0,\infty)} \le \alpha^{-1} \|\gamma\|_{L_1(0,\infty)}. \tag{3.15}$$

Proof. Statement (3.13) is a consequence of the fact that a convolution of two functions which belong to classes L_p and L_q, respectively, where $1 \le p \le \infty$ and $1/p + 1/q = 1$, is a continuous function of t (cf. [10, p. 216]).

Statement (3.14) follows immediately from the estimates

$$|g(t)| \leq e^{-\alpha t/2} \int_0^{t/2} |\gamma(\tau)|\, d\tau + \int_{t/2}^{t} |\gamma(\tau)|\, d\tau$$
$$\leq e^{-\alpha t/2} \int_0^{\infty} |\gamma(\tau)|\, d\tau + \int_{t/2}^{\infty} |\gamma(\tau)|\, d\tau.$$

In order to show (3.15), observe that

$$\|g\|_{L_1(0,\infty)} = \int_0^{\infty} \left| \int_0^t e^{-\alpha(t-\tau)} \gamma(\tau)\, d\tau \right| dt \leq \int_0^{\infty} \left(\int_0^t e^{-\alpha(t-\tau)} |\gamma(\tau)|\, d\tau \right) dt.$$

Changing the order of integration in the last integral, we get

$$\int_0^{\infty} \left(\int_0^t e^{-\alpha(t-\tau)} |\gamma(\tau)|\, d\tau \right) dt = \int_0^{\infty} \left(\int_{\tau}^{\infty} e^{-\alpha(t-\tau)} |\gamma(\tau)|\, dt \right) d\tau$$
$$= \alpha^{-1} \|\gamma\|_{L_1(0,\infty)}.$$

This completes the proof of the lemma.

The following lemma is useful in the proof of statement (ii) of Theorem 2.

LEMMA 2. *Let* $\gamma \in L_1(0,\infty)$ *be a positive function and let* α, δ, η *be positive constants. If* $\delta < \alpha\eta$, *then there exists a positive continuous function* h *on* $[0,\infty)$ *with the following properties*:

$$h(t) \to 0 \quad \text{as } t \to \infty, \tag{3.16}$$

$$h \in L_1(0,\infty), \tag{3.17}$$

$$\int_0^t e^{-\alpha(t-\tau)} [\delta h(\tau) + \gamma(\tau)]\, d\tau = \eta h(t) \quad \text{for all } t \geq 0. \tag{3.18}$$

Proof. For $h \in L_1(0,\infty)$, we define

$$\mathcal{F}h(t) = \eta^{-1} \int_0^t e^{-\alpha(t-\tau)} [\delta h(\tau) + \gamma(\tau)]\, d\tau, \qquad t \geq 0.$$

By Lemma 1, $\mathcal{F}h$ is continuous on $[0,\infty)$, $\mathcal{F}h(t) \to 0$ as $t \to \infty$ and $\mathcal{F}h \in L_1(0,\infty)$. Thus, $\mathcal{F}$ maps $L_1(0,\infty)$ into itself. The proof is complete if we show that operator $\mathcal{F}$ has a fixed point h which is positive on $[0,\infty)$.

For $h_1, h_2 \in L_1(0,\infty)$ and $t \geq 0$, we have

$$\left|\mathscr{F}h_1(t) - \mathscr{F}h_2(t)\right| \leq \eta^{-1}\delta\int_0^t e^{-\alpha(t-\tau)}\left|h_1(\tau) - h_2(\tau)\right| d\tau,$$

which, by Lemma 1 (cf. (3.15)), implies

$$\|\mathscr{F}h_1 - \mathscr{F}h_2\|_{L_1(0,\infty)} \leq \eta^{-1}\delta\alpha^{-1}\|h_1 - h_2\|_{L_1(0,\infty)}.$$

Since $\delta < \alpha\eta$, $\mathscr{F}$: $L_1(0,\infty) \to L_1(0,\infty)$ is a contraction mapping and it has a unique fixed point $h \in L_1(0,\infty)$. It remains to show that h is positive on $[0,\infty)$.

It is known that the fixed point h of operator $\mathscr{F}$ can be written as a limit of successive approximations

$$h = \lim_{\nu\to\infty} h_\nu \quad \text{in } L_1(0,\infty), \tag{3.19}$$

where $h_0 \in L_1(0,\infty)$ is arbitrary and $h_{\nu+1} = \mathscr{F}h_\nu$, $\nu = 0, 1, \dots$. Taking $h_0 \equiv 0$, it can be seen by easy induction that

$$h_\nu(t) \geq \eta^{-1}\int_0^t e^{-\alpha(t-\tau)}\gamma(\tau)\,d\tau, \qquad t \geq 0,\ \nu = 1, 2, \dots . \tag{3.20}$$

From (3.19), it follows that $h(t) = \lim_{\nu\to\infty} h_\nu(t)$ for almost every $t \in [0,\infty)$. Consequently, letting $\nu \to \infty$ in (3.20), we obtain

$$h(t) \geq \eta^{-1}\int_0^t e^{-\alpha(t-\tau)}\gamma(\tau)\,d\tau \tag{3.21}$$

for almost every $t \in [0,\infty)$. Since $h = \mathscr{F}h$ is continuous on $[0,\infty)$, (3.21) holds for all $t \in [0,\infty)$ and the proof is complete.

Proof of Theorem 2. Let $\sigma \geq 0$, $\phi \in C$ be arbitrary. By the variation-of-constants formula, the solution $z = z(\sigma,\phi)$ of Eq. (2.3) can be written as

$$z_t = y_t + \int_\sigma^t U(t,\tau)X_0 f(\tau, z_\tau)\,d\tau, \qquad t \geq \sigma, \tag{3.22}$$

where $y = y(\sigma,\phi)$ is the solution of Eq. (2.2) and $U(t,\tau)$: $C \to C$ is the solution operator of Eq. (2.2) defined by

$$U(t,\tau)\phi = y_t(\tau,\phi), \qquad t \geq \tau \geq 0,\ \phi \in C.$$

According to the proof of Theorem 1 (cf. (3.12)), there exists a constant $\varkappa > 0$ such that

$$\begin{aligned} \|y_t\| &\leq \varkappa\|\phi\|, && t \geq \sigma, \\ \left\|U(t,\tau)X_0\right\| &\leq \varkappa, && t \geq \tau \geq 0. \end{aligned} \tag{3.23}$$

By the triangle inequality, we have

$$\begin{aligned}|f(\tau, z_\tau)| &= |f(\tau,0) + f(\tau, z_\tau) - f(\tau,0)| \\ &\le |f(\tau,0)| + |f(\tau, z_\tau) - f(\tau,0)|,\end{aligned}$$

which, together with (2.5), implies

$$|f(\tau, z_\tau)| \le |f(\tau,0)| + \gamma(\tau)\|z_\tau\|, \qquad \tau \ge \sigma. \tag{3.24}$$

From (3.22), in view of (3.23) and (3.24), it follows that

$$\|z_t\| \le \varkappa\|\phi\| + \varkappa\int_0^\infty |f(\tau,0)|\,d\tau + \varkappa\int_0^t \gamma(\tau)\|z_\tau\|\,d\tau, \qquad t \ge \sigma,$$

which, by the Gronwall inequality, yields

$$\|z_t\| \le \varkappa\left[\|\phi\| + \int_0^\infty |f(\tau,0)|\,d\tau\right]\exp\left(\varkappa\int_0^\infty \gamma(\tau)\,d\tau\right)$$

for $t \ge \sigma$. Thus, z is bounded on $[\sigma, \infty)$. This, together with (2.8), (2.9), and (3.24), implies

$$\int_0^\infty |f(\tau, z_\tau)|\,d\tau < \infty. \tag{3.25}$$

By Theorem 1, the limits

$$\psi = \lim_{t\to\infty} y_t,$$

and

$$\overline{U}(\tau) = \lim_{t\to\infty} U(t,\tau)X_0, \qquad \tau \ge 0 \tag{3.26}$$

exist in C and $C([-r,0], \mathbb{R}^{n^2})$, respectively. Relations (3.23), (3.25), and (3.26) show that we can apply Lemma 6 of [4] to the components of the integral in (3.22), which implies that

$$\int_\sigma^t U(t,\tau)X_0 f(\tau, z_\tau)\,d\tau \to \int_\sigma^\infty \overline{U}(\tau) f(\tau, z_\tau)\,d\tau \quad \text{as } t\to\infty,$$

the last integral being absolutely convergent. Therefore (cf. (3.22)):

$$z_t \to \psi_* \stackrel{\text{def}}{=} \psi + \int_\sigma^\infty \overline{U}(\tau) f(\tau, z_\tau)\,d\tau \quad \text{as } t\to\infty.$$

Since, according to Theorem 1, ψ and the columns of $\overline{U}(\tau)$, $\tau \ge \sigma$, belong to P, ψ_* also belongs to P. That is, $\psi_* = \phi_\xi$ for some $\xi \in E$. Clearly, $z(t) \to \xi$ as $t \to \infty$ which completes the proof of statement (i).

Now we prove statement (ii). Let $\xi \in E$ be given. Denote by B the vector space of continuous functions z: $[\sigma, \infty) \to C$ such that (writing z_t instead of $z(t)$),

$$\|z\|_B \stackrel{\text{def}}{=} \sup_{t \geq \sigma} \|z_t^P\| + \sup_{t \geq \sigma} \left[\frac{1}{h(t)} \|z_t^Q\| \right] < \infty, \tag{3.27}$$

where h is a positive continuous function on $[\sigma, \infty)$ which is specified later. It is easy to show that $\|\cdot\|_B$ is a norm on B and $(B, \|\cdot\|_B)$ is a Banach space.

On B, (using the notation from the proof of Theorem 1) we define an operator $\mathscr{K}$ by

$$(\mathscr{K}z)_t = \phi_\xi - \int_t^\infty T(t-\tau) X_0^P [M(\tau, z_\tau) + f(\tau, z_\tau)] \, d\tau$$
$$+ \int_\sigma^t T(t-\tau) X_0^Q [M(\tau, z_\tau) + f(\tau, z_\tau)] \, d\tau$$

for $t \geq \sigma$. In view of (3.6), $(\mathscr{K}z)_t$ can be written as

$$(\mathscr{K}z)_t = \phi_\xi - \int_t^\infty T(t-\tau) X_0^P [M(\tau, z_\tau^Q) + f(\tau, z_\tau)] \, d\tau$$
$$+ \int_\sigma^t T(t-\tau) X_0^Q [M(\tau, z_\tau^Q) + f(\tau, z_\tau)] \, d\tau.$$

The projections of $(\mathscr{K}z)_t$ onto subspaces P and Q have the form

$$(\mathscr{K}z)_t^P = \phi_\xi - \int_t^\infty T(t-\tau) X_0^P [M(\tau, z_\tau^Q) + f(\tau, z_\tau)] \, d\tau, \tag{3.28a}$$

$$(\mathscr{K}z)_t^Q = \int_\sigma^t T(t-\tau) X_0^Q [M(\tau, z_\tau^Q) + f(\tau, z_\tau)] \, d\tau. \tag{3.28b}$$

By the definition of the norm $\|\cdot\|_B$, we have

$$\begin{aligned} &\|z_t^P\| \leq \|z\|_B, \\ &\|z_t^Q\| \leq h(t) \|z\|_B, \\ &\|z_t\| \leq [1 + h(t)] \|z\|_B, \qquad t \geq \sigma, \end{aligned} \tag{3.29}$$

the last inequality being a consequence of the first and second one, since $\|z_t\| = \|z_t^P + z_t^Q\| \leq \|z_t^P\| + \|z_t^Q\|$. From (3.28b), in view of (2.4), (3.4), (3.24), and (3.29), we obtain

$$\left\|(\mathscr{K}z)_t^Q\right\| \leq K\int_\sigma^t e^{-\alpha(t-\tau)}\{\mu(\tau)h(\tau)\|z\|_B + |f(\tau,0)| + \gamma(\tau)[1+h(\tau)]\|z\|_B\}\,d\tau,$$

and hence

$$\left\|(\mathscr{K}z)_t^Q\right\| \leq K(1+\|z\|_B) \times \int_\sigma^t e^{-\alpha(t-\tau)}\{\mu(\tau)h(\tau) + |f(\tau,0)| + \gamma(\tau)[1+h(\tau)]\}\,d\tau \tag{3.30}$$

for $t \geq \sigma$.

Let δ be an arbitrary constant such that $0 < \delta < \frac{\alpha}{2K}$. By Lemma 2, there exists a positive continuous function h on $[0,\infty)$ with the following properties,

$$h(t) \to 0 \quad \text{as } t \to \infty, \tag{3.31}$$

$$h \in L_1(0,\infty), \tag{3.32}$$

$$\int_0^t e^{-\alpha(t-\tau)}\{\delta h(\tau) + |f(\tau,0)| + 2\gamma(\tau)\}\,d\tau = \frac{1}{2K}h(t) \quad \text{for all } t \geq 0. \tag{3.33}$$

Choose $\sigma_0 \geq 0$ such that

$$\sup_{t\geq\sigma_0} \mu(t) < \delta, \tag{3.34}$$

and

$$\sup_{t\geq\sigma_0} h(t) < 1. \tag{3.35}$$

(The existence of σ_0 follows from (2.7) and (3.31).) Then (3.30) and (3.31) imply that

$$\sup_{t\geq\sigma}\left[\frac{1}{h(t)}\left\|(\mathscr{K}z)_t^Q\right\|\right] \leq \frac{1}{2}(1+\|z\|_B) \tag{3.36}$$

provided $\sigma \geq \sigma_0$.

From (3.28a), by similar estimates as before, we get

$$\left\| (\mathscr{K}z)_t^P \right\| \le |\xi| + K\int_t^\infty \{ \mu(\tau) h(\tau) \|z\|_B + |f(\tau,0)| + \gamma(\tau)[1 + h(\tau)] \|z\|_B \} \, d\tau.$$

Hence (cf. (3.34) and (3.35)),

$$\sup_{t \ge \sigma} \left\| (\mathscr{K}z)_t^P \right\| \le |\xi| + K(1 + \|z\|_B) \int_\sigma^\infty \{ \delta h(\tau) + |f(\tau,0)| + 2\gamma(\tau) \} \, d\tau \tag{3.37}$$

provided $\sigma \ge \sigma_0$.

From (3.36) and (3.37), we see that if h and σ_0 are chosen as before and $\sigma \ge \sigma_0$, then operator $\mathscr{K}$ is well defined and maps B into itself.

Let $z_1, z_2 \in B$. By similar estimates as in the proof of (3.36) and (3.37), we obtain

$$\sup_{t \ge \sigma} \left[\frac{1}{h(t)} \left\| (\mathscr{K}z_1 - \mathscr{K}z_2)_t^Q \right\| \right] \le \frac{1}{2} \|z_1 - z_2\|_B, \tag{3.38}$$

and

$$\sup_{t \ge \sigma} \left\| (\mathscr{K}z_1 - \mathscr{K}z_2)_t^P \right\| \le K \left[\delta \int_\sigma^\infty h(\tau) \, d\tau + 2\int_\sigma^\infty \gamma(\tau) \, d\tau \right] \|z_1 - z_2\|_B \tag{3.39}$$

provided $\sigma \ge \sigma_0$. Let $\sigma \ge \sigma_0$ be chosen such that

$$\delta \int_\sigma^\infty h(\tau) \, d\tau + 2\int_\sigma^\infty \gamma(\tau) \, d\tau < \frac{1}{3K}.$$

(Such constant certainly exists.) Then (3.38) and (3.39) imply that $\|\mathscr{K}z_1 - \mathscr{K}z_2\|_B \le \frac{5}{6}\|z_1 - z_2\|_B$ for all $z_1, z_2 \in B$. Thus, $\mathscr{K}\colon B \to B$ is a contraction mapping. It is easily seen that the unique fixed point $z \in B$ of operator $\mathscr{K}$ is a solution of Eq. (2.3) such that $\|z_t - \phi_\xi\| \to 0$ as $t \to \infty$. The proof of the theorem is complete.

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