JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **191**, 547–571 (1995)

Linear Theory of Abstract Functional Differential Equations of Retarded Type

OVIDE ARINO

Université de Pau, U.R.A.-C.N.R.S. 1204, 64000 Pau, France

AND

EVA SANCHEZ

ETS Ingenieros Industriales, Universidad Politécnica de Madrid, Madrid, Spain

Submitted by G. F. Webb

Received December 16, 1993

The purpose of this paper is to provide an extension of the linear theory of functional differential equations of retarded type to abstract equations. Such equations include examples borrowed from population dynamics to which the theory applies. An application will be given elsewhere. Our main effort in this work consists in providing a suitable extension of the formal adjoint equation and the formal duality. The solutions of the linear autonomous retarded functional differential equation

$$x'(t) = L(x_t), \tag{1}$$

where L is a bounded linear operator mapping the space $C([-r, 0]; E)$ into the Banach space E, define a strongly continuous translation semigroup. We show the existence of a direct sum decomposition of $C([-r, 0]; E)$ into two subspaces which are semigroup invariants. The flow induced by the solutions of Eq. (1) can be interpreted as the flow induced by an ordinary differential equation in a finite-dimensional space. We explicitly characterize this decomposition by an orthogonality relation associated to a certain definition of formal duality. The existence of an integral representation for the operator L leads to an equation formally adjoint to (1) characterizing the projection operator defined by the above decomposition of $C([-r, 0]; E)$. © 1995 Academic Press, Inc.

0022-247X/95 $6.00

INTRODUCTION

Let us consider the Cauchy problem for the linear autonomous retarded functional differential equation

$$\begin{aligned} x'(t) &= L(x_t), \qquad t > 0 \\ x_0 &= \varphi, \end{aligned} \tag{1}$$

where $L: C([-r, 0]; E) \to E$ $(r > 0)$ is a bounded linear operator and E is a Banach space. The solution of (1) is a function $x \in C([-r, +\infty[; E)$, $x \in C^1([0, +\infty[; E)$, which satisfies (1) for $t > 0$. As usual, we denote by x_t the section at t of the function x, $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

We know already from [8] that the unique solution of (1), which exists for each initial value $\varphi \in C([-r, 0]; E)$, is associated with a strongly continuous semigroup of translations. More precisely, we have

THEOREM 1. *The operator $Af = \dot{f}$ with domain*

$$D(A) = \{f \in C^1([-r, 0]; E); \dot{f}(0) = L(f)\}$$

is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $C([-r, 0]; E)$ satisfying the translation property

$$T(t)f(\theta) = \begin{cases} f(t + \theta) & \text{if } t + \theta \leq 0 \\ T(t + \theta)f(0) & \text{if } t + \theta > 0, \end{cases}$$

$t > 0$, $\theta \in [-r, 0]$, $f \in C([-r, 0]; E)$.

Furthermore, for each $\varphi \in C([-r, 0]; E)$, define $x: [-r, +\infty[\to E$ by

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0] \\ T(t)\varphi(0) & \text{if } t > 0. \end{cases}$$

Then x is the unique solution of (1) *and $T(t)\varphi = x_t$, $t > 0$.*

Let $\sigma(A)$, $\sigma_e(A)$ be the spectrum and the essential spectrum [20], respectively, of the infinitesimal generator A. From the general theory about operator reduction for isolated points of the spectrum, we obtain the following theorem, which yields the decomposition of $C([-r, 0]; E)$ into a direct sum:

THEOREM 2. *Let $\lambda \in \sigma(A) - \sigma_e(A)$. Then λ is an eigenvalue of A and for some positive integer m we have*

$$C([-r, 0]; E) = \mathcal{N}(A - \lambda I)^m \oplus \mathcal{R}(A - \lambda I)^m,$$

where $\mathcal{N}(A - \lambda I)^m$ *is the generalized eigenspace of A with respect to* λ *and* $\dim \mathcal{N}(A - \lambda I)^m = q < +\infty$.

Moreover A is completely reduced by this decomposition, A restricted to $\mathcal{N}(A - \lambda I)^m$ *is bounded with spectrum* $\{\lambda\}$, *and the subspaces* $\mathcal{N}(A - \lambda I)^m, \mathcal{R}(A - \lambda I)^m$ *are invariant under the semigroup* $\{T(t)\}_{t\geq 0}$ [10, 24].

The results concerning the behaviour of solutions obtained by J. Hale [9] for the problem (1) in finite-dimensional spaces remain valid without essential modifications. Let $\Phi_\lambda = (\varphi_1, \ldots, \varphi_q)$ be a basis for $\mathcal{N}(A - \lambda I)^m$. There is a $q \times q$ constant matrix B_λ such that $\dot{\Phi}_\lambda = \Phi_\lambda B_\lambda$ and λ is the unique eigenvalue of B_λ. Therefore

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{B_\lambda \theta}, \qquad \theta \in [-r, 0]$$

and also

$$T(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t}, \qquad t > 0.$$

Furthermore, if the initial value φ of (1) belongs to $\mathcal{N}(A - \lambda I)^m$, we have $\varphi = \Phi_\lambda a$ for some q-vector a and the solution is defined by

$$x_t = T(t)\varphi = T(t)\Phi_\lambda a = \Phi_\lambda e^{B_\lambda t}a, \qquad t > 0.$$

The same theory applies for a finite subset of $\sigma(A) - \sigma_e(A)$ and gives a very clear description of the geometric behaviour of the solutions of (1). We summarize these results in the following theorem.

THEOREM 3. *Suppose* $\Lambda = \{\lambda_1 \ldots \lambda_s\}$ *is any finite subset of the non-essential spectrum of A,* $\sigma(A) - \sigma_e(A)$, *and let* $\Phi_\Lambda = (\Phi_{\lambda_1}, \ldots, \Phi_{\lambda_s})$, $B_\Lambda = \operatorname{diag}(B_{\lambda_1}, \ldots, B_{\lambda_s})$, *where* Φ_{λ_j} *is a basis of the generalized eigenspace associated to* λ_j, $\mathcal{N}(A - \lambda_j I)^{m_j}$, *with* $\dim \mathcal{N}(A - \lambda_j I)^{m_j} = q_j < +\infty$, *and* B_{λ_j} *is a constant matrix such that* $A\Phi_{\lambda_j} = \Phi_{\lambda_j}B_{\lambda_j}$. *The only eigenvalue of* B_{λ_j} *is* λ_j, $j = 1, \ldots, s$. *Moreover, let*

$$P_\Lambda = \mathcal{N}(A - \lambda_1 I)^{m_1} \oplus \cdots \oplus \mathcal{N}(A - \lambda_s I)^{m_s}.$$

Then there exists a subspace Q_Λ *of* $C([-r, 0]; E)$ *invariant under A and* $\{T(t)\}_{t\geq 0}$, *such that*

$$C([-r, 0]; E) = P_\Lambda \oplus Q_\Lambda$$

and the operator A is completely reduced by this decomposition.

Furthermore, for any initial value $\varphi = \Phi_\Lambda a$, *where a is a constant vector of dimension* $q_1 + \cdots + q_s$, *the solution of the Cauchy problem* (1) *is defined by*

$$x_t = T(t)\varphi = T(t)\Phi_\Lambda a = \Phi_\Lambda e^{B_\Lambda t} a, \qquad t \geq 0.$$

We say that A *is reduced by* Λ.

In this paper we obtain an explicit characterization for the projection operator on the subspace Q_Λ. To do this, we define a *formal duality* as a certain bilinear form which allows us to state a Fredholm alternative theorem.

It is known that any bounded linear operator L: $C([-r, 0]; E) \to E$ is represented by an integral associated with a bounded semivariation vector measure. From this we define an operator A^* *formally adjoint* to A which helps in characterizing Q_Λ by means of the orthogonality relationship with respect to formal duality.

There have been several attempts to extend the theory of linear functional differential equations from finite to infinite dimensions. Most of them are motivated by the study of partial differential equations with delay. A seminal work on that, and a classic reference, is a paper by Travis and Webb [23]. A short discussion of our work compared to that of others together with prospective work along these lines is given in the conclusion of the paper.

1. Formal Duality and a Fredholm Alternative Theorem

Let us consider $C([0, r]; E^*)$, where E^* is the topological dual space of E. We shall define a continuous bilinear form denoted $\langle\langle \alpha, \varphi \rangle\rangle$ on the product $C([0, r]; E^*) \times C([-r, 0]; E)$ which will be interpreted as a *formal duality*.

A function f: $[0, r] \to E^*$ is called *simple* if there exist two finite collections $x_1^*, \ldots, x_p^* \in E^*$ and $A_1, \ldots, A_p \in \Sigma$ with $\bigcup_{i=1}^p A_i = [0, r]$, $A_i \cap A_j = \varnothing$ such that

$$f = \sum_{i=1}^{p} x_i^* \chi_{A_i},$$

where χ_A is the characteristic function of A and Σ is the Borel algebra on $[0, r]$. Denote $S([0, r]; E^*)$ the space of simple functions.

Definition 1. For any $\alpha \in S([0, r]; E^*)$ and $\varphi \in C([0, r]; E)$ we define the bilinear form

$$\langle\langle \alpha, \varphi \rangle\rangle = \langle \alpha(0), \varphi(0) \rangle + \sum_{i=1}^{p} \left\langle x_i^*, L\left(\int_\theta^0 \chi_{A_i}(\xi - \theta)\varphi(\xi)\, d\xi\right)\right\rangle,$$

where $\alpha = \sum_{i=1}^{p} x_i^* \chi_{A_i}$ and $\langle *, * \rangle$ denote the usual duality between E^* and E.

The value $\langle\langle \alpha, \varphi \rangle\rangle$ is independent of the representation chosen for α as a linear combination of characteristic functions.

Since

$$|\langle\langle \alpha, \varphi \rangle\rangle| \leq (1 + r\|L\|)\|\alpha\| \, \|\varphi\|$$

there exists a unique continuous extension of this bilinear form to the completion of $S([0, r]; E^*) \times C([-r, 0]; E)$, where both spaces are equipped with the sup norm.

We restrict our extension to the product $C([0, r]; E^*) \times C([-r, 0]; E)$ and we call this the *formal duality* associated with the operator L.

It is interesting to specify the formal duality for $\alpha \in C([0, r]) \otimes E^*$.

LEMMA 1. *Let $f \in C([0, r])$ and $u^* \in E^*$. We consider the function $f \otimes u^* \in C([0, r]; E^*)$ defined by $(f \otimes u^*)(s) = f(s)u^*$, $s \in [0, r]$. Then*

$$\langle\langle f \otimes u^*, \varphi \rangle\rangle = \langle u^*, f(0)\varphi(0) \rangle + \left\langle u^*, L\left(\int_\theta^0 f(\xi - \theta)\varphi(\xi)\, d\xi\right)\right\rangle.$$

Proof. The function f is representable as the limit of a uniformly convergent sequence of simple real functions defined in $[0, r]$,

$$f = \lim_{n \to \infty} \sum_{i=1}^{p_n} \beta_i^{(n)} \chi_{A_i^{(n)}},$$

and therefore the sequence of functions in $S([0, r]; E^*)$,

$$\left\{ \sum_{i=1}^{p_n} \beta_i^{(n)} u^* \otimes \chi_{A_i^{(n)}} \right\}_{n=1,2,\ldots},$$

converges to $f \otimes u^*$.

By continuity of the formal duality, we obtain

$$\begin{aligned}
&\langle\langle f \otimes u^*, \varphi \rangle\rangle \\
&\quad = \langle (f \otimes u^*)(0), \varphi(0) \rangle + \lim_{n \to \infty} \sum_{i=1}^{p_n} \left\langle \beta_i^{(n)} u^*, L\left(\int_\theta^0 \chi_{A_i^{(n)}}(\xi - \theta)\varphi(\xi)\, d\xi\right)\right\rangle \\
&\quad = \langle (f \otimes u^*)(0), \varphi(0) \rangle + \lim_{n \to \infty} \sum_{i=1}^{p_n} \left\langle u^*, L\left(\int_\theta^0 \beta_i^{(n)} \chi_{A_i^{(n)}}(\xi - \theta)\varphi(\xi)\, d\xi\right)\right\rangle \\
&\quad = \langle u^*, f(0)\varphi(0) \rangle + \langle u^*, L\left(\int_\theta^0 f(\xi - \theta)\varphi(\xi)\, d\xi\right)\rangle.
\end{aligned}$$

In particular for $\varepsilon_\lambda \otimes u^*$ with $\varepsilon_\lambda(\theta) = e^{\lambda\theta}$ we have

$$\langle\langle \varepsilon \otimes u^*, \varphi\rangle\rangle = \langle u^*, \varphi(0)\rangle + \left\langle u^*, L\left(\int_\theta^0 e^{\lambda(\xi-\theta)}\varphi(\xi)\, d\xi\right)\right\rangle.$$

We shall characterize the closed subspace $\mathcal{R}(A - \lambda I)$ with λ in $\sigma(A) - \sigma_e(A)$, in terms of the formal duality and to do this we need to reduce $\sigma(A)$ to the spectrum of some operators defined on E. More precisely, for $\lambda \in \mathbf{C}$ we define the operator $L_\lambda \in \mathcal{L}(E)$ by

$$L_\lambda(u) = L(\varepsilon_\lambda \otimes u), \qquad u \in E.$$

As usual, $\mathcal{L}(E)$ is the space of bounded linear operators on E.

We know that $\lambda \in \sigma(A)$ if and only if $\lambda \in \sigma(L_\lambda)$. Also, the operator R_λ: $C([-r, 0]; E) \to E$ defined by

$$R_\lambda(\varphi) = \varphi(0) + L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\varphi(\xi)\, d\xi\right)$$

is onto [8].

PROPOSITION 1. *Let $\lambda \in \sigma(A) - \sigma_e(A)$. Then $\varphi \in \mathcal{R}(A - \lambda I)$ if and only if $\langle\langle \alpha, \varphi\rangle\rangle = 0$ for all $\alpha = \varepsilon_{-\lambda} \otimes x^*$, with $x^* \in \mathcal{N}(L_\lambda^* - \lambda I)$, where L_λ^* is the adjoint operator of L_λ.*

Proof. We need two auxiliary results stated in the following lemmas and whose proofs are postponed until the Appendix.

LEMMA 2. *For any $\lambda \in \mathbf{C}$ we have $\varphi \in \mathcal{R}(A - \lambda I)$ if and only if $R_\lambda(\varphi) \in \mathcal{R}(L_\lambda - \lambda I)$.*

LEMMA 3. *The subspace $\mathcal{R}(A - \lambda I)$ is closed in $C([-r, 0]; E)$ if and only if the subspace $\mathcal{R}(L_\lambda - \lambda I)$ is closed in E.*

Since $\lambda \in \sigma(A) - \sigma_e(A)$, we conclude from Lemma 3 that $\mathcal{R}(L_\lambda - \lambda I)$ is closed in E and then $R_\lambda(\varphi) \in \mathcal{R}(L_\lambda - \lambda I)$ if and only if $\langle x^*, R_\lambda(\varphi)\rangle = 0$ for all $x^* \in \mathcal{N}(L_\lambda^* - \lambda I)$. But $\langle x^*, R_\lambda(\varphi)\rangle = \langle\langle \varepsilon_{-\lambda} \otimes x^*, \varphi\rangle\rangle$.

We can state Proposition 1 in alternative Fredholm theorem form:

Let $\lambda \in \sigma(A) - \sigma_e(A)$. Then, the equation $(A - \lambda I)\psi = \varphi$ has a solution if and only if $\langle\langle \varepsilon_{-\lambda} \otimes x^*, \varphi\rangle\rangle = 0$ for all $x^* \in \mathcal{N}(L_\lambda^* - \lambda I)$.

2. CHARACTERIZATION OF THE SUBSPACE $\mathcal{R}(A - \lambda I)^m$ WITH λ IN $\sigma(A) - \sigma_e(A)$

Our approach is based on the characterization of the subspace $\mathcal{R}(A - \lambda I)^m$ in terms of another operator defined on product space E^m. If the equation $(A - \lambda I)^m \varphi = \psi$ is to have a solution, then

$$\varphi(\theta) = \sum_{j=0}^{m-1} u_j \frac{\theta^j}{j!} e^{\lambda\theta} + \int_0^\theta e^{\lambda(\theta-\xi)} \frac{(\theta - \xi)^{m-1}}{(m-1)!} \psi(\xi)\, d\xi, \qquad \theta \in [-r, 0],$$

where $u_0, \ldots, u_{m-1}$ are arbitrary elements of E which must be determined so that $\varphi \in D(A - \lambda I)^m$.

Introducing the notation

$$\varphi^{(k)}(\theta) = \left(\frac{d}{d\theta} - \lambda I\right)^k \varphi(\theta), \qquad \theta \in [-r, 0],$$

we have $\varphi \in D(A - \lambda I)^m$ if and only if $\varphi^{(k)} \in D(A)$, $k = 0, \ldots, m - 1$. By direct calculation it is easy to obtain the condition

$$\mathcal{L}_\lambda^{(m)}(u_0, \ldots, u_{m-1})^{\mathrm{T}} = \mathcal{R}_\lambda^{(m)}(\psi),$$

where $(\ldots)^{\mathrm{T}}$ means the transpose vector and we have introduced the operators

$$\mathcal{L}_\lambda^{(m)} \in \mathcal{L}(E^m); \qquad \mathcal{L}_\lambda^{(m)} = \begin{bmatrix} \lambda I - L_\lambda & I - L_\lambda^1 & \ldots & -L_\lambda^{m-2} & -L_\lambda^{m-1} \\ 0 & \lambda I - L_\lambda & \ldots & -L_\lambda^{m-3} & -L_\lambda^{m-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda I - L_\lambda & I - L_\lambda^1 \\ 0 & 0 & \ldots & 0 & \lambda I - L_\lambda \end{bmatrix}$$

with $L_\lambda^j \in \mathcal{L}(E)$, $L_\lambda^j(u) = L(\varepsilon_\lambda^j \otimes u)$, $u \in E$, and

$$\varepsilon_\lambda^j(\theta) = \frac{\theta^j}{j!} e^{\lambda\theta}, \qquad j = 1, \ldots, m - 1$$

and also $\mathcal{R}_\lambda^{(m)}: C([-r, 0]; E) \to E$ is defined by

$$\mathcal{R}_\lambda^{(m)}(\psi) = \begin{bmatrix} L(\int_0^\theta e^{\lambda(\theta-\xi)}((\theta - \xi)^{m-1}/(m-1)!)\, \psi(\xi)\, d\xi) \\ L(\int_0^\theta e^{\lambda(\theta-\xi)}((\theta - \xi)^{m-2}/(m-2)!)\, \psi(\xi)\, d\xi) \\ \vdots \\ L(\int_0^\theta e^{\lambda(\theta-\xi)}(\theta - \xi)\psi(\xi)\, d\xi) \\ -\psi(0) + L(\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)\, d\xi) \end{bmatrix}.$$

Therefore we have proved the following result

LEMMA 4. *$\psi \in \mathcal{R}(A - \lambda I)^m$ if and only if $\mathcal{R}_\lambda^{(m)}(\psi) \in \mathcal{R}(\mathcal{L}_\lambda^{(m)})$.*

We wish to locate $\mathcal{R}(\mathcal{L}_\lambda^{(m)})$ in terms of the adjoint operator $\mathcal{L}_\lambda^{(m)*}$. In order to achieve this, we need to see that $\mathcal{R}(\mathcal{L}_\lambda^{(m)})$ is a closed subspace of E^m. No other result similar to the one stated in Lemma 3 is known.

Since $\mathcal{R}(A - \lambda I)^m = (\mathcal{R}_\lambda^{(m)})^{-1}(\mathcal{R}(\mathcal{L}_\lambda^{(m)}))$, one obtains that if $\mathcal{R}(\mathcal{L}_\lambda^{(m)})$ is a closed subspace in E^m, then $\mathcal{R}(A - \lambda I)^m$ is closed in $C([-r, 0]; E)$. However, the converse statement is more interesting to us. We begin its analysis proving the following proposition.

PROPOSITION 2. *Let $\lambda \in \sigma(A) - \sigma_e(A)$. Then $\mathcal{L}_\lambda^{(m)}$ is a Fredholm operator for each $m = 1, 2, \ldots$.*

The proof is an immediate consequence of the two next lemmas, which are proved in the Appendix.

LEMMA 5. *If $\lambda \in \sigma(A) - \sigma_e(A)$, then $L_\lambda - \lambda I$ is a Fredholm operator.*

LEMMA 6. *Let $A_1 \ldots A_m$ be Fredholm operators on E. The operator $\mathcal{A}^{(m)}$ defined on the product space E^m by*

$$\mathcal{A}^{(m)} = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$

where the operators $*$ are in $\mathcal{L}(E)$, is a Fredholm operator.

We return to the problem set before the statement of Proposition 2 and we conclude that $\mathcal{R}_\lambda^{(m)}(\psi) \in \mathcal{R}(\mathcal{L}_\lambda^{(m)})$ if and only if $\langle X^*, \mathcal{R}_\lambda^{(m)}(\psi)\rangle = 0$ for all $X^* = (x_0^*, \ldots, x_{m-1}^*)^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)*})$.

Note that

$$\langle\langle \hat{\varepsilon}_\lambda^j \otimes u^*, \varphi\rangle\rangle = \langle u^*, \varepsilon_\lambda^j(0)\varphi(0)\rangle + \left\langle u^*, L\left(\int_\theta^0 \varepsilon_\lambda^j(\theta - \xi)\varphi(\xi)\, d\xi\right)\right\rangle,$$

where

$$\hat{\varepsilon}_\lambda^j(s) = \varepsilon_\lambda^j(-s) = \frac{(-s)^j}{j!} e^{-\lambda s}$$

and therefore

$$\langle X^*, \mathcal{R}_\lambda^{(m)}(\psi)\rangle = \left\langle\left\langle \sum_{j=0}^{m-1} \hat{\varepsilon}_\lambda^j \otimes x_{m-j-1}^*, \psi\right\rangle\right\rangle = 0.$$

We consider the subspace

$$\mathcal{H}_\lambda^{(m)*} = \left\{\alpha = \sum_{j=0}^{m-1} \hat{\varepsilon}_\lambda^j \otimes x_{m-j-1}^*;\ (x_0^*, \ldots, x_{m-1}^*)^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)*})\right\}.$$

Since $\mathcal{L}_\lambda^{(m)}$ is a Fredholm operator, we have $\dim \mathcal{N}(\mathcal{L}_\lambda^{(m)*}) = p$ for some $p < +\infty$ and as well $\dim \mathcal{H}_\lambda^{(m)*} = p$.

We summarize the previous work in the following proposition.

PROPOSITION 3. *Let $\lambda \in \sigma(A) - \sigma_e(A)$ and let m be a positive integer. Then $\psi \in \mathcal{R}(A - \lambda I)^m$ if and only if $\langle\langle \alpha, \psi \rangle\rangle = 0$ for all $\alpha \in \mathcal{H}_\lambda^{(m)*}$.*

Also, $\varphi \in \mathcal{N}(A - \lambda I)^m$ means $(A - \lambda I)^m \varphi = 0$, with $\varphi \in D(A - \lambda I)^m$ and then

$$\mathcal{N}(A - \lambda I)^m = \left\{\varphi = \sum_{j=0}^{m-1} \varepsilon_\lambda^j \otimes u_j;\ (u_0, \ldots, u_{m-1})^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)})\right\}$$

with $\dim \mathcal{N}(A - \lambda I)^m = \dim \mathcal{N}(\mathcal{L}_\lambda^{(m)}) < +\infty$.

Note that $(x_0^*, \ldots, x_{m-1}^*)^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)*})$ implies also that $(0, x_0^*, \ldots, x_{m-2}^*)^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)*})$ and then it is easy to prove by direct calculation that the subspace $\mathcal{H}_\lambda^{(m)*}$ is differentiation invariant.

This fact implies that elements of this subspace are solutions of a linear ODE. In fact, choosing a basis $\Phi_\lambda^* = (\varphi_1^*, \ldots, \varphi_p^*)^{\mathrm{T}}$ of $\mathcal{H}_\lambda^{(m)*}$, we have

$$\dot{\Phi}_\lambda^* = (\dot{\varphi}_1^*, \ldots, \dot{\varphi}_p^*)^{\mathrm{T}} = B_\lambda^*(\varphi_1^*, \ldots, \varphi_p^*)^{\mathrm{T}} = B_\lambda^* \Phi_\lambda^*,$$

where B_λ^* is a constant $p \times p$ matrix and λ is the only eigenvalue of this matrix. Therefore

$$\Phi_\lambda^*(\theta) = e^{B_\lambda^* \theta} \Phi_\lambda^*(0), \qquad \theta \in [0, r].$$

3. DECOMPOSITION OF THE SPACE $C([-r, 0]; E)$

We have seen that for $\lambda \in \sigma(A) - \sigma_e(A)$ there exists a positive integer m for which a direct sum decomposition of the following type is verified,

$$C([-r, 0]; E) = \mathcal{N}(A - \lambda I)^m \oplus \mathcal{R}(A - \lambda I)^m$$

with $\dim \mathcal{N}(A - \lambda I)^m = q < +\infty$ and $\varphi \in \mathcal{R}(A - \lambda I)^m$ if and only if $\langle\langle \alpha, \varphi \rangle\rangle = 0$ for all $\alpha \in \mathcal{H}_\lambda^{(m)*}$. Also, $\dim \mathcal{H}_\lambda^{(m)*} = p < +\infty$.

We wish to find a suitable coordinate system which serves in characterizing the projection operator on the subspace $\mathcal{R}(A - \lambda I)^m$. To this end, we shall find a relationship between the numbers p, q.

Recall that $p = \dim \mathcal{N}(\mathcal{L}_\lambda^{(m)*})$, $q = \dim \mathcal{N}(\mathcal{L}_\lambda^{(m)})$ but in general, even for Fredholm operators, we have $p \neq q$. In the following, we state that $p \leq q$ and we obtain some sufficient conditions for $p = q$.

Let $\Psi_\lambda = (\psi_1, \ldots, \psi_q)$ be a basis for the subspace $\mathcal{N}(A - \lambda I)^m$ and $\Psi_\lambda^* = (\alpha_1^*, \ldots, \alpha_p^*)^{\mathrm{T}}$ another one for $\mathcal{K}_\lambda^{(m)*}$, and make up the constant $p \times q$ matrix

$$M = [\langle\langle \alpha_i^*, \psi_j \rangle\rangle]_{i,j=1,\ldots,p,q} = \langle\langle \Psi_\lambda^*, \Psi_\lambda \rangle\rangle.$$

If $(\lambda_1, \ldots, \lambda_q)^{\mathrm{T}} \in \mathcal{N}(M)$, then $\langle\langle \alpha^*, \lambda_1\psi_1 + \cdots + \lambda_q\psi_q \rangle\rangle = 0$ for all α^* in $\mathcal{K}_\lambda^{(m)*}$ and Proposition 3 implies that $\lambda_1\psi_1 + \cdots + \lambda_q\psi_q \in \mathcal{R}(A - \lambda I)^m$. But also, $\lambda_1\psi_1 + \cdots + \lambda_q\psi_q \in \mathcal{N}(A - \lambda I)^m$ and then $\lambda_1\psi_1 + \cdots + \lambda_q\psi_q = 0$. Therefore $\lambda_i = 0$, $i = 1, \ldots, q$, $\mathcal{N}(M) = \{0\}$. Thus we conclude that M is of rank q, implying that $q \leq p$.

In particular we can choose two new bases, $\Phi_\lambda = (\varphi_1, \ldots, \varphi_q)$, $\Phi_\lambda^* = (\varphi_1^*, \ldots, \varphi_p^*)^{\mathrm{T}}$, such that the constant $p \times q$ matrix satisfies

$$\langle\langle \Phi_\lambda^*, \Phi_\lambda \rangle\rangle = [\delta_{ij}]_{i,j=1,\ldots,p,q},$$

where δ_{ij} is the Kronecker symbol.

We summarize all of this in a theorem, stating in this way one of the main results in this paper:

THEOREM 4. *If* $\lambda \in \sigma(A) - \sigma_e(A)$ *then* $\dim \mathcal{N}(A - \lambda I)^m \leq \dim \mathcal{K}_\lambda^{(m)*}$ *and there exist two bases,* $\Phi_\lambda = (\varphi_1, \ldots, \varphi_q)$, $\Phi_\lambda^* = (\varphi_1^*, \ldots, \varphi_p^*)^{\mathrm{T}}$, *of the subspaces* $\mathcal{N}(A - \lambda I)^m$ *and* $\mathcal{K}_\lambda^{(m)*}$, *respectively, such that the* $p \times q$ *constant matrix* $\langle\langle \Phi_\lambda^*, \Phi_\lambda \rangle\rangle = [\langle\langle \varphi_i^*, \varphi_j \rangle\rangle]_{i,j=1,\ldots,p,q}$ *satisfies* $\langle\langle \Phi_\lambda^*, \Phi_\lambda \rangle\rangle = [\delta_{ij}]_{i,j=1,\ldots,p,q}$.

Moreover, for each $\varphi \in C([-r, 0]; E)$ *we have a unique decomposition* $\varphi = \varphi_K + \varphi_I$ *with* $\varphi_K \in \mathcal{N}(A - \lambda I)^m$, $\varphi_I \in \mathcal{R}(A - \lambda I)^m$, *and* $\langle\langle \varphi_j^*, \varphi_I \rangle\rangle = 0$, $j = 1, \ldots, p$.

Also, $\varphi_K = \sum_{i=1}^q \lambda_i\varphi_i$ *with* $\langle\langle \varphi_i^*, \varphi_K \rangle\rangle = \lambda_i$ *if* $i \leq q$ *and* $\langle\langle \varphi_i^*, \varphi_K \rangle\rangle = 0$ *if* $i > q$.

Next we relate the matrices B_λ, B_λ^* defined by $\dot{\Phi}_\lambda = \Phi_\lambda B_\lambda$, $\dot{\Phi}_\lambda^* = B_\lambda^* \Phi_\lambda^*$.

It is easy to check that for $\alpha = f \otimes u^*$, $f \in C^1([0, r])$, $u^* \in E^*$, and for all $\varphi \in D(A)$, we have

$$\langle\langle \alpha, \dot{\varphi} \rangle\rangle + \langle\langle \dot{\alpha}, \varphi \rangle\rangle = \langle u^*, L(\hat{f} \otimes \varphi(0)) \rangle + \langle u^*, f(0)\varphi(0) \rangle,$$

where $\hat{f}(\theta) = f(-\theta)$.

Thus, for $\alpha \in \mathcal{K}_\lambda^{(m)*}$ and $\varphi \in D(A)$,

$$\langle\langle \alpha, \dot{\varphi} \rangle\rangle + \langle\langle \dot{\alpha}, \varphi \rangle\rangle = 0.$$

Therefore

$$\langle\langle \dot{\Phi}^*_\lambda, \Phi_\lambda \rangle\rangle = B^*_\lambda \langle\langle \Phi^*_\lambda, \Phi_\lambda \rangle\rangle = -\langle\langle \Phi^*_\lambda, \dot{\Phi}_\lambda \rangle\rangle$$
$$= -\langle\langle \Phi^*_\lambda, \Phi_\lambda \rangle\rangle B_\lambda .$$

Since

$$\langle\langle \Phi^*_\lambda, \Phi_\lambda \rangle\rangle = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$$

with I_q the identity matrix of order q, we can obtain

$$B^*_\lambda = \begin{bmatrix} -B_\lambda & N \\ 0 & P \end{bmatrix},$$

where N, P are two matrices of required dimensions and 0 is a zero submatrix.

Note that for $p = q$ the last relation reduces to

$$B^*_\lambda = -B_\lambda .$$

Now we state some sufficient conditions to obtain $p = q$.

LEMMA 7. *If the formal duality is non-degenerate then $p = q$.*

Proof. We say, as usual, that the formal duality *is non-degenerate* if the equality $\langle\langle \alpha, \varphi \rangle\rangle = 0$ for all $\varphi \in C([-r, 0]; E)$ implies that $\alpha = 0$.

With the above notations, let $(\mu_1, \ldots, \mu_p)^T \in \mathcal{N}(M^T)$. Then $\langle\langle \mu_1 \alpha_1^* + \cdots + \mu_p \alpha_p^*, \varphi \rangle\rangle = 0$ for all $\varphi \in \mathcal{R}(A - \lambda I)^m$ and also for all $\varphi \in C([-r, 0]; E)$. Since the formal duality is non-degenerate, $\mu_1 \alpha_1^* + \cdots + \mu_p \alpha_p^*$ must be equal to zero and then $\mathcal{N}(M^T) = \{0\}$. This implies that $p \leq q$.

The converse is not true in general. There are examples for finite-dimensional spaces E such that $p = q$ and the formal duality is degenerate.

Finally, we relate the equality $p = q$ with some compactness properties on the operator $\mathcal{L}^{(m)}_\lambda$. Introduce the operators

$$J^{(m)} = \begin{bmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \qquad H^{(m)}_\lambda = \begin{bmatrix} L_\lambda & L^1_\lambda & \ldots & L^{m-2}_\lambda & L^{m-1}_\lambda \\ 0 & L_\lambda & \ldots & L^{m-3}_\lambda & L^{m-2}_\lambda \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & L_\lambda & L^1_\lambda \\ 0 & 0 & \ldots & 0 & L_\lambda \end{bmatrix}.$$

If $\lambda \neq 0$ the operator $\lambda I + J^{(m)}$ is invertible and then

$$\mathcal{L}_\lambda^{(m)} = \lambda I + J^{(m)} - H_\lambda^{(m)} = (\lambda I + J^{(m)})(I - (\lambda I + J^{(m)})^{-1} H_\lambda^{(m)}).$$

The operators $J^{(m)}$ and $H_\lambda^{(m)}$ commute and from well known general results about the spectrum of compact operators [18, Th. 4.25] we get to

LEMMA 8. *Let $\lambda \in \sigma(A) - \sigma_e(A)$ be an eigenvalue of A, $\lambda \neq 0$. If the operator $H_\lambda^{(m)}$ or some of its iterates are compact then $p = q$.*

LEMMA 9. *Let λ, μ be given in $\sigma(A) - \sigma_e(A)$, $\lambda \neq \mu$. For any $m, r \in \mathbf{N}^*$, $\alpha \in \mathcal{K}_\lambda^{(m)*}$, $\varphi \in \mathcal{N}(A - \mu I)^r$ we have $\langle\langle \alpha, \varphi \rangle\rangle = 0$.*

Proof. Given that the two polynomials in x, $(x - \lambda)^m$, and $(x - \mu)^r$ are relatively prime, we infer from the Bezout identity the existence of two polynomials $P(x)$, $Q(x)$ such that

$$I = \left(\frac{d}{d\theta} - \lambda I\right)^m P\left(\frac{d}{d\theta}\right) + \left(\frac{d}{d\theta} - \mu I\right)^r Q\left(\frac{d}{d\theta}\right),$$

where $d/d\theta$ is the differentiation operator, so that

$$\begin{aligned}\langle\langle \alpha, \varphi \rangle\rangle &= \left\langle\left\langle \alpha, \left(\frac{d}{d\theta} - \lambda I\right)^m P\left(\frac{d}{d\theta}\right)\varphi \right\rangle\right\rangle + \left\langle\left\langle \alpha, \left(\frac{d}{d\theta} - \mu I\right)^r Q\left(\frac{d}{d\theta}\right)\varphi \right\rangle\right\rangle \\ &= \left\langle\left\langle \left(-\frac{d}{d\theta} - \lambda I\right)^m \alpha, P\left(\frac{d}{d\theta}\right)\varphi \right\rangle\right\rangle + \left\langle\left\langle \alpha, Q\left(\frac{d}{d\theta}\right)\left(\frac{d}{d\theta} - \mu I\right)^r \varphi \right\rangle\right\rangle \\ &= 0.\end{aligned}$$

Let $\Lambda = \{\lambda_1, \ldots, \lambda_s\}$ be a finite subset of non-essential points of $\sigma(A)$ and consider the decomposition of the space $C([-r, 0]; E)$ stated in Theorem 3. We are now able to characterize the subspace Q_Λ by an orthogonality relation associated with the formal duality.

To this end, we define

$$P_\Lambda^* = \mathcal{K}_{\lambda_1}^{(m_1)*} \oplus \cdots \oplus \mathcal{K}_{\lambda_s}^{(m_s)*}.$$

Next, let Φ_j, Φ_j^* be bases of the subspaces $\mathcal{N}(A - \lambda_j I)^{m_j}$, $\mathcal{K}_{\lambda_j}^{(m_j)*}$, respectively, $j = 1, \ldots, s$. From the results stated above we know that each constant $p_j \times q_j$ matrix $J_j = \langle\langle \Phi_j^*, \Phi_j \rangle\rangle$, $q_j \leq p_j$, has rank equal to q_j and also that the matrix $\langle\langle \Phi_k^*, \Phi_l \rangle\rangle$ is equal to zero for $k \neq l$. Then the matrix J of order $(p_1 + \cdots + p_s) \times (q_1 + \cdots + q_s)$,

$$J = [\langle\langle \Phi_j^*, \Phi_k \rangle\rangle]_{j,k=1,\ldots,s},$$

has rank equal to $q_1 + \cdots + q_s$.

Therefore there exist two bases Φ_Λ, Φ_Λ^* of the subspaces P_Λ, P_Λ^*, respectively, such that the constant matrix $\langle\langle \Phi_\Lambda^*, \Phi_\Lambda >$ satisfies

$$\langle\langle \Phi_\Lambda^*, \Phi_\Lambda \rangle\rangle = [\delta_{ij}]_{i,j=1,\ldots,p_1+\cdots+p_s,\, q_1+\cdots+q_s}.$$

Finally, we have found the characterization of the projection onto the subspace Q_Λ. We retain all the above notations to state this last result as follows:

THEOREM 5. *Consider the direct sum decomposition*

$$C([-r, 0]; E) = P_\Lambda \oplus Q_\Lambda.$$

Then,

$$Q_\Lambda = \{\varphi \in C([-r, 0]; E); \langle\langle \Phi_\Lambda^*, \varphi \rangle\rangle = 0\}.$$

Moreover, any $\varphi \in C([-r, 0]; E)$ *may be written as* $\varphi = \varphi_{P_\Lambda} + \varphi_{Q_\Lambda}$ *with* $\langle\langle \Phi_\Lambda^*, \varphi_{Q_\Lambda} \rangle\rangle = 0$ *and* $\varphi_{P_\Lambda} = \Phi_\Lambda a$, *where a is a constant vector of dimension* $q_1 + \cdots + q_s$ *such that*

$$\langle\langle \Phi_\Lambda^*, \varphi \rangle\rangle = \langle\langle \Phi_\Lambda^*, \varphi_{P_\Lambda} \rangle = \langle\langle \Phi_\Lambda^*, \Phi_\Lambda \rangle\, a = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

and 0 *is the zero vector of dimension* $p_1 + \cdots + p_s - (q_1 + \cdots + q_s)$.

Note that if $p_j = q_j$, $j = 1, \ldots, s$, then

$$\varphi_{P_\Lambda} = \Phi_\Lambda a = \Phi_\Lambda \langle\langle \Phi_\Lambda^*, \varphi \rangle\rangle.$$

4. THE FORMAL ADJOINT EQUATION

Before proceeding to the construction of this equation let us recall some well known results about the integral representation of bounded linear operators defined on $C([-r, 0]; E)$. We refer to [6] for the general theory.

Any bounded linear operator $L: C([-r, 0]; E) \to E$ determines a unique vector measure $m: \hat{\Sigma} \to \mathcal{L}(E; E^{**})$ of bounded semivariation and such that for all $f \in C([-r, 0]; E)$ we have

$$L(f) = \int_{[-r,0]} f\, dm,$$

where $\mathscr{L}(E; E^{**})$ is the space of the bounded linear operators defined on E with values in E^{**}, where E^{**} is the bidual of E, and $\hat{\Sigma}$ is the Borel algebra on $[-r, 0]$.

For each $x^* \in E^*$ there exists a vector measure $m_{x^*}: \hat{\Sigma} \to E^*$ defined by

$$\langle m_{x^*}(A), x\rangle = \langle x^*, m(A)(x)\rangle; \qquad A \in \hat{\Sigma},\ x \in E,$$

which satisfies

$$\int_{[-r,0]} f\, dm_{x^*} = \langle x^*, L(f)\rangle; \qquad \forall x^* \in E^*.$$

Next we define the linear operator $\tilde{L}: S([0, r]; E^*) \to E^*$ for any $f \in S([0, r]; E^*)$, $f = \sum_{i=1}^{r} x_i^* \chi_{A_i}$ by

$$\tilde{L}(f) = \sum_{i=1}^{r} m_{x_i^*}(-A_i).$$

LEMMA 10. *If the vector measure m is of bounded variation, then $\tilde{L}$ is continuous with respect to the sup norm in $S([0, r]; E^*)$.*

Proof. Let $f = \sum_{i=1}^{r} x_i^* \chi_{A_i}$ be a simple function with $\|x_i^*\| \leq 1$, $i = 1, \ldots, r$. Then

$$\|\tilde{L}(f)\| = \left\|\sum_{i=1}^{r} m_{x_i^*}(-A_i)\right\| \leq \sum_{i=1}^{r} \|m_{x_i^*}(-A_i)\|$$

$$\leq \sum_{i=1}^{r} \|m(-A_i)\| \leq v(m)([-r, 0]) < +\infty,$$

where $v(m)$ means the variation of m and we have used

$$\|m_{x_i^*}(-A_i)\| = \sup_{\|x\|\leq 1} |\langle m_{x_i^*}(-A_i), x\rangle|$$

$$= \sup_{\|x\|\leq 1} |\langle x_i^*, m(-A_i)(x)\rangle| \leq \|m(-A_i)\|.$$

Under this hypothesis, there exists a unique continuous extension $\tilde{\tilde{L}}$ of the operator $\tilde{L}$ to the completion of $S([0, r]; E^*)$ equipped with the sup norm and we are able to define the *formal adjoint operator* of the operator L.

DEFINITION 2. The operator L^* is the restriction to the space $C([0, r]; E^*)$ of the extension operator $\tilde{L}$.

Just as in the case of the formal duality, it is convenient to obtain the expression of L^* for the elements in $C([0, r]) \otimes E^*$. Calculations very similar to those above for Lemma 1 show that

LEMMA 11. *For each* $f \otimes u^* \in C([0, r]) \otimes E^*$, $L^*(f \otimes u^*) \in E^*$ *is the linear form defined by*

$$\langle L^*(f \otimes u^*), u\rangle = \langle u^*, L(\hat{f} \otimes u)\rangle; \qquad u \in E.$$

In particular, for the functions ε_λ^j introduced above, we reach the result

$$\begin{aligned}\langle L^*(\hat{\varepsilon}_\lambda^j \otimes u^*), u\rangle &= \langle u^*, L(\varepsilon_\lambda^j \otimes u)\rangle \\ &= \langle u^*, L_\lambda^j(u)\rangle = \langle L_\lambda^{j^*}(u^*), u\rangle; \qquad u \in E\end{aligned}$$

and then

$$L^*(\hat{\varepsilon}_\lambda^j \otimes u^*) = L_\lambda^{j^*}(u^*).$$

The existence of the operator L^* allows us to define a new linear functional differential equation associated with problem (1).

DEFINITION 3. The formal adjoint equation associated to (1) is

$$\dot{\alpha}(s) = -L^*(\alpha_s); \qquad s \leq 0. \tag{2}$$

A function $\alpha \in C(]-\infty, r]; E^*)$ is a solution if $\alpha \in C^1(]-\infty, 0]; E^*)$ and satisfies (2) for all $s \leq 0$.

It is easy to check that $\alpha(s) = e^{-\lambda s}x^*$, $s \leq 0$, is a solution of (2) for all $x^* \in \mathcal{N}(L_\lambda^* - \lambda I)$. Suppose that $\alpha(t) = f(t)x^*$ is a solution of (2) on $]-\infty, b]$ and that $x(t)$ is a solution of Eq. (1) on $[a, +\infty[$, $a < b$. Then $\langle\langle \alpha_t, x_t\rangle\rangle$ is constant for all $t \in [a, b]$.

Indeed

$$\begin{aligned}\langle\langle \alpha_t, x_t\rangle\rangle &= \langle \alpha_t(0), x_t(0)\rangle + \left\langle x^*, L\left(\int_\theta^0 f_t(\xi - \theta)x_t(\xi)\, d\xi\right)\right\rangle \\ &= \langle \alpha(t), x(t)\rangle + \left\langle x^*, L\left(\int_{t+\theta}^t f(\omega - \theta)x(\omega)\, d\omega\right)\right\rangle.\end{aligned}$$

Therefore

$$\frac{d}{dt}\langle\langle \alpha_t, x_t\rangle\rangle = \langle \dot\alpha(t), x(t)\rangle + \langle \alpha(t), \dot x(t)\rangle$$
$$+ \langle x^*, L(f(t-\theta)x(t))\rangle - \langle x^*, L(f(t)x(t+\theta))\rangle.$$

But since

$$\langle \alpha(t), \dot x(t)\rangle - \langle x^*, L(f(t)x(t+\theta))\rangle$$
$$= \langle \alpha(t), \dot x(t)\rangle - \langle \alpha(t), L(x_t)\rangle = 0$$

we have

$$\frac{d}{dt}\langle\langle \alpha_t, x_t\rangle\rangle = \langle \dot\alpha(t), x(t)\rangle + \langle x^*, L(f(t-\theta)x(t))\rangle$$
$$= \langle \dot\alpha(t), x(t)\rangle + L^*(f_t \otimes x^*), x(t)\rangle$$
$$= \langle \dot\alpha(t), x(t)\rangle + \langle L^*(\alpha_t), x(t)\rangle = 0.$$

5. The Operator A^* Formal Adjoint of A

In the sequel we accept that the vector measure m associated to L is of bounded variation and L^* has been defined.

Definition 4. We call the formal adjoint operator of A relative to the formal duality the operator A^*, defined by

$$A^*(\alpha) = -\dot\alpha; \qquad D(A^*) = \{\alpha \in C^1([0, r]; E^*); \dot\alpha(0) = -L^*(\alpha)\}.$$

A^* is linear and closed, with domain dense in $C([0, r]; E^*)$.

From Lemma 1 and after an adequate integration by parts we obtain for $\alpha = f \otimes x^* \in D(A^*)$

$$\langle\langle \alpha, A\varphi\rangle\rangle = \langle\langle A^*\alpha, \varphi\rangle\rangle; \qquad \forall\varphi \in D(A).$$

Proposition 4. $\sigma(A) = \sigma(A^*)$.

Proof. The solution of $(A^* - \lambda I)\varphi^* = \psi^*$ with $\psi^* \in C([0, r]; E^*)$ is

$$\varphi^*(\theta) = e^{-\lambda\theta}\varphi^*(0) - \int_0^\theta e^{\lambda(s-\theta)}\psi^*(s)\, ds; \qquad \theta \in [0, r],$$

where $\varphi^*(0)$ is to be determined so that $\varphi^* \in D(A^*)$. We get to

$$(L_\lambda^* - \lambda I)(\varphi^*(0)) = \psi^*(0) + L^*\left(\int_0^\theta e^{\lambda(s-\theta)}\psi^*(s)\,ds\right).$$

Since the operator $R_\lambda^*: C([0, r]; E^*) \to E^*$, defined by

$$R_\lambda^*(\psi^*) = \psi^*(0) + L^*\left(\int_0^\theta e^{\lambda(s-\theta)}\psi^*(s)\,ds\right),$$

is onto, we conclude that $\lambda \in \sigma(A^*)$ if and only if $\lambda \in \sigma(L_\lambda^*)$.

We have seen above that $\lambda \in \sigma(A)$ if and only if $\lambda \in \sigma(L_\lambda)$ and it is well known that $\sigma(L_\lambda) = \sigma(L_\lambda^*)$ [22]. Therefore the result follows.

Note that

$$\mathcal{N}(A - \lambda I) = \{\varepsilon_\lambda \otimes x;\ x \in \mathcal{N}(L_\lambda - \lambda I)\}$$
$$\mathcal{N}(A^* - \mu I) = \{\varepsilon_{-\mu} \otimes x^*;\ x^* \in \mathcal{N}(L_\mu^* - \mu I)\}.$$

If λ, μ are eigenvalues of A, A^*, respectively, and $\lambda \neq \mu$, then $\langle\langle \alpha, \varphi \rangle\rangle = 0$ for all $\alpha \in \mathcal{N}(A^* - \mu I)$, $\varphi \in \mathcal{N}(A - \lambda I)$.

Indeed

$$\langle\langle A^*\alpha, \varphi \rangle\rangle = \mu\langle\langle \alpha, \varphi \rangle\rangle = \langle\langle \alpha, A\varphi \rangle\rangle = \lambda\langle\langle \alpha, \varphi \rangle\rangle$$

but $\lambda \neq \mu$ implies that $\langle\langle \alpha, \varphi \rangle\rangle = 0$.

This enables us to state again Proposition 1 in a Fredholm alternative theorem form.

PROPOSITION 5. *Let $\lambda \in \sigma(A) - \sigma_e(A)$. Then, the equation $(A - \lambda I)\varphi = \psi$ has a solution if and only if $\langle\langle \alpha, \psi \rangle\rangle = 0$ for all solutions α of the equation $(A^* - \lambda I)\alpha = 0$.*

We try to obtain a direct sum decomposition of $C([0, r]; E^*)$, which plays for A^* the same role as the one stated for the space $C([-r, 0]; E)$ in Theorem 3. In a similar way we shall define a generalized eigenspace of the formal adjoint operator A^*, associated with a finite collection $\Lambda = \{\lambda_1, \ldots, \lambda_s\} \subset \sigma(A) - \sigma_e(A)$, which helps us to characterize the subspace Q_Λ of Theorem 5 in terms of the orthogonality associated with the formal duality. From the following lemmas we can easily obtain the equality between the non-essential spectra of A and A^*.

LEMMA 12. *The subspace $\mathcal{R}(A^* - \lambda I)$ is closed in $C([0, r]; E^*)$ if and only if $\mathcal{R}(A - \lambda I)$ is closed in $C([-r, 0]; E)$.*

Proof. Arguments similar to those developed in Lemmas 2 and 3 lead to

(a) If we define the operator $R_\lambda^*: C([0, r]; E^*) \to E^*$ by

$$R_\lambda^*(\psi) = \psi(0) + L^* \left(\int_0^\theta e^{\lambda(s-\theta)} \psi(s)\, ds \right)$$

then, for any $\lambda \in \mathbf{C}$, we have $\alpha \in \mathcal{R}(A^* - \lambda I)$ if and only if $R_\lambda^*(\alpha) \in \mathcal{R}(L_\lambda^* - \lambda I)$.

(b) The subspace $\mathcal{R}(A^* - \lambda I)$ is closed in $C([0, r]; E^*)$ if and only if $\mathcal{R}(L_\lambda^* - \lambda I)$ is closed in E^*.

Now we finish by recalling the known result that $\mathcal{R}(L_\lambda - \lambda I)$ is a closed subspace in E if and only if $\mathcal{R}(L_\lambda^* - \lambda I)$ is closed in E^*.

LEMMA 13. *For any $\lambda \in \mathbf{C}$ and $m = 1, 2, \ldots$ we have*

$$\mathcal{N}(A^* - \lambda I)^m = \mathcal{K}_\lambda^{(m)*}.$$

Proof. The equation $(A^* - \lambda I)^m \alpha = 0$ has a solution

$$\alpha(\theta) = \sum_{j=0}^{m-1} u_{m-1-j}^* \frac{(-\theta)^j}{j!} e^{-\lambda\theta}; \qquad \theta \in [0, r]$$

with $(u_0^*, \ldots, u_{m-1}^*)$ such that $\alpha \in D(A^* - \lambda I)^m$.

By setting up the notation

$$\alpha^{(k)} = (A^* - \lambda I)^k \alpha = \sum_{j=0}^{m-1-k} \hat{\varepsilon}_\lambda^j \otimes u_{m-1-k-j}^*$$

and making calculations like those in Lemma 4 above, we see that $\alpha \in \mathcal{N}(A^* - \lambda I)^m$ if and only if $\alpha = \sum_{j=0}^{m-1} \hat{\varepsilon}_\lambda^j \otimes u_{m-1-j}^*$ with $(u_0^*, \ldots, u_{m-1}^*)^{\mathrm{T}} \in \mathcal{N}(\mathcal{L}_\lambda^{(m)*})$. But this is the characterization of the space $\mathcal{K}_\lambda^{(m)*}$.

Last, let $\Lambda = \{\lambda_1, \ldots, \lambda_s\}$ be a finite subset of non-essential points of $\sigma(A)$ and let $\mathcal{N}(A^* - \lambda_j I)^{m_j}$, $j = 1, \ldots, s$, be the generalized eigenspaces associated to these eigenvalues, with $\dim \mathcal{N}(A^* - \lambda_j I)^{m_j} = p_j < +\infty$. The results obtained so far allow us to state the existence of a direct sum decomposition for the space $C([0, r]; E^*)$ which reduces the operator A^*, similar to the one established for the space $C([-r, 0]; E)$ in Theorem 5.

Then, there exists a subspace $Q_\Lambda{}^*$ invariant under A^* such that

$$C([0, r]; E^*) = P_\Lambda^* \oplus Q_\Lambda^*,$$

where

$$P_\Lambda^* = \mathcal{N}(A^* - \lambda_1 I)^{m_1} \oplus \cdots \oplus \mathcal{N}(A^* - \lambda_s I)^{m_s}.$$

6. Conclusion

Several extensions of Hale's theory of functional differential equations to infinite dimensions exist in the literature, starting from the one given by Travis and Webb [23]. The main motivation is the study of partial differential equations with finite or infinite delay. In most cases (in works of Kunish and Schappacher [12, 13] and Schumacher [21], to name a few), the equation is of the form

$$x'(t) = Ax(t) + \int_{-r}^{0} d\eta(s)x(t + s),$$

where A generates a c_0-semigroup on a Banach space X and $d\eta$ is a suitable restricted Stieltjes measure with values in $\mathcal{L}(X)$. Another important motivation is related to control theory. Increasingly elaborate extensions of earlier work by Bernier and Manitius [5] and Manitius [14] on attainability, completeness, or degeneracy, to the case of partial differential equations with delays have been given by Nakagiri [15–17]. The works by Nakagiri include results on the spectral theory of such equations and the characterization of some generalized eigenspaces in terms of the solutions of an adjoint equation. Many of the considerations of this author are similar to ours. The results differ in the following ways: some restrictions are imposed by Nakagiri [16, 17] on the measure, which takes the form

$$\int_{-r}^{0} d\eta(s)\varphi(s) = \sum A_i\varphi(-\tau_i) + \int_{-r}^{0} D(s)\varphi(s)\, ds$$

with $A_i \in \mathcal{L}(X)$, $D \in L^1((-r, 0); \mathcal{L}(X))$. Moreover, it is assumed that the space X is reflexive. In contrast, we make no hypotheses on the functional term or on the space. Let us finally describe briefly the problem which motivated us to build a general theory. Our interest stems from the study of an equation of cell population dynamics introduced first by Kimmel *et al.* [11]. The equation reads

$$n(t, x) = \int_{0}^{+\infty} g(x, y)n(t - \theta(y), y)\, dy, \tag{3}$$

where g and θ are given and satisfy appropriate assumptions [2], and n is the state variable. Equation (3) sets up a difference-integral equation which leads to a strongly continuous positive semigroup on a space $L^1(\Omega)$, $\Omega \subset (-r, 0) \times \mathbf{R}^+$. Under additional conditions on θ and g, one can associate with Eq. (3) an abstract delay differential equation of type

$$u'(t) = L(u_t),$$

where $u(t) = n(t, .) \in X := L^1(\mathbf{R}^+)$, and

$$(L\varphi) = \int\int k(x, y, z)\varphi(-\theta(y) - \theta(z), z)\, dy\, dz.$$

Equation (3) and some variants of it have been the subject of several studies by Arino and Kimmel [2, 3], Sanchez, Arino, and Kimmel [19], and others. Most of the results concentrate on the determination of the so-called asynchronous exponential growth [1]. In [2–4], nonlinear perturbations of Eq. (3) are considered. It is our hope that the framework provided by the theory of linear abstract delay differential equations will make it easier to study dynamical properties of nonlinear perturbations of (3), especially those arising from changes of stability of steady states, Hopf bifurcation, etc. Basic tools for these purposes are spectral theory of non-essential eigenvalues, projectors onto associated invariant subspaces, which possibly behave nicely with respect to time, and a good variation of constants formula. These are precisely the extent to which we have aimed at expanding our study of abstract delay differential equations. While the present article has been devoted to describing spectral decomposition for non-essential eigenvalues and the formal dual product and the adjoint equation associated with it, the derivation of a variation of constants formula and the application to the study of Eq. (3) will be presented elsewhere.

APPENDIX: PROOFS OF LEMMAS

Proof of Lemma 2. $\varphi \in \mathcal{R}(A - \lambda I)$ if and only if there exists $\psi \in D(A)$ such that $(A - \lambda I)\psi = \varphi$. This differential equation has the solution

$$\psi(\theta) = e^{\lambda\theta}\psi(0) - \int_\theta^0 e^{\lambda(\theta-\xi)}\varphi(\xi)\, d\xi, \qquad \theta \in [-r, 0],$$

where $\psi \in D(A)$ and then $\dot{\psi}(0) = L(\psi)$,

$$\varphi(0) + \lambda\psi(0) = L(\varepsilon_\lambda \otimes \psi(0)) - L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\varphi(\xi)\, d\xi\right);$$

that is,

$$(L_\lambda - \lambda I)(\psi(0)) = \varphi(0) + L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\varphi(\xi)\, d\xi\right) = R_\lambda(\varphi).$$

The solution of the last equation exists if and only if $R_\lambda(\varphi) \in \mathcal{R}(L_\lambda - \lambda I)$.

Proof of Lemma 3. First, we suppose that $\mathcal{R}(L_\lambda - \lambda I)$ is a closed subspace of E and let $\{\psi_n\} \subset \mathcal{R}(A - \lambda I)$ be a convergent sequence with $\psi = \lim_{n\to\infty} \psi_n$.

From Lemma 2 we have $R_\lambda(\psi_n) \in \mathcal{R}(L_\lambda - \lambda I)$ and then $R_\lambda(\psi) \in \mathcal{R}(L_\lambda - \lambda I)$. Again Lemma 2 enables us to conclude that $\psi \in \mathcal{R}(A - \lambda I)$ and so $\mathcal{R}(A - \lambda I)$ is closed in $C([-r, 0]; E)$.

Now we shall demonstrate the converse. To this end we recall that R_λ is surjective and the equation $R_\lambda(\varphi) = b$ has a solution of the form

$$\varphi(\xi) = \Phi_k(\xi)(I + T_k)^{-1}(b),$$

where k is some positive integer satisfying the conditions $\Phi_k \in C([-r, 0])$, $\operatorname{supp} \Phi_k \subset [-1/k, 0]$, $\Phi_k(0) = 1$, $0 \le \Phi_k(\theta) \le 1$. The operator $T_k \in \mathcal{L}(E)$ is defined by

$$T_k(b) = L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\Phi_k(\xi) b\, d\xi\right).$$

Indeed, it is enough to observe that $\lim_{k\to\infty} \|T_k\| = 0$ and then for some k, the operator $I + T_k$ has an inverse.

We return to the proof of the converse. Let $\{v_n\} \subset \mathcal{R}(L_\lambda - \lambda I)$ be a convergent sequence with $v = \lim_{n\to\infty} v_n$. For each n the equation $R_\lambda(\tilde{\psi}_n) = v_n$ has a solution

$$\tilde{\psi}_n(\xi) = \Phi_k(\xi)(I + T_k)^{-1}(v_n).$$

It is easy to see that $\{\tilde{\psi}_n\} \subset \mathcal{R}(A - \lambda I)$ is a Cauchy sequence in $C([-r, 0]; E)$. Then there exist $\tilde{\psi} = \lim_{n\to\infty} \tilde{\psi}_n$ and $\tilde{\psi} \in \overline{\mathcal{R}(A - \lambda I)}$. Therefore

$$R_\lambda(\tilde{\psi}) = \lim_{n\to\infty} R_\lambda(\tilde{\psi}_n) = \lim_{n\to\infty} v_n = v.$$

If $\mathcal{R}(A - \lambda I)$ is a closed subspace in $C([-r, 0]; E)$ and $v \in \overline{\mathcal{R}(L_\lambda - \lambda I)}$, we have seen that there exists $\tilde{\psi} \in \overline{\mathcal{R}(A - \lambda I)} = \mathcal{R}(A - \lambda I)$ such that $v = R_\lambda(\tilde{\psi})$. Now, from Lemma 2 we conclude that $v \in \mathcal{R}(L_\lambda - \lambda I)$.

Proof of Lemma 5. Since $\lambda \in \sigma(A) - \sigma_e(A)$, from Lemma 3 we infer that $\mathcal{R}(L_\lambda - \lambda I)$ is a closed subspace in E. Moreover $\mathcal{N}(A - \lambda I)$ is made by the functions $\varepsilon_\lambda \otimes u$ with $u \in \mathcal{N}(L_\lambda - \lambda I)$ and then $\dim \mathcal{N}(L_\lambda - \lambda I) < +\infty$.

It is enough to prove that $\mathcal{R}(L_\lambda - \lambda I)$ is of finite codimension.

Let M be a finite-dimensional complementary subspace of $\mathcal{R}(A - \lambda I)$, let P_M be the projection on M, and let Φ_k be one of the functions defined in the proof of Lemma 3. Denote by $\Phi_k \otimes E$ the space of the functions $\Phi_k \otimes b$ with $b \in E$. The finite-dimensional subspace $P_M(\Phi_k \otimes E)$ admits a basis

$$\{P_M(\Phi_k \otimes e_1), \ldots, P_M(\Phi_k \otimes e_r)\},$$

where $\{e_1, \ldots, e_r\}$ is a fixed finite subset in E. For each $b \in E$ we can write

$$P_M(\Phi_k \otimes b) = \sum_{i=1}^{r} \alpha_i P_M(\Phi_k \otimes e_i).$$

The scalars $\alpha_1, \ldots, \alpha_r$ are determined in terms of b and then the operator

$$S(b) = \sum_{i=1}^{r} \alpha_i e_i; \qquad b \in E$$

is well defined. Also, $S \in \mathcal{L}(E)$.

Finally, we consider the operator $T \in \mathcal{L}(E)$ defined by

$$T(b) = R_\lambda(\Phi_k \otimes (b - S(b)); \qquad b \in E.$$

Introduce the notations

$$K_1(b) = b - L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\Phi_k(\xi) b\, d\xi\right)$$

$$K_2(b) = -S(b) + L\left(\int_\theta^0 e^{\lambda(\theta-\xi)}\Phi_k(\xi) S(b)\, d\xi\right).$$

Since K_1 is an isomorphism for adequate k and K_2 is a finite rank operator, $K_2 K_1^{-1}$ is a compact operator. But

$$T(b) = (I + K_2 K_1^{-1})(K_1(b)); \qquad b \in E$$

and therefore T is of Fredholm [18] and [7]. Then $\mathcal{R}(T)$ is a closed subspace of finite codimension in E.

Lemma 5 follows from $\mathcal{R}(T) \subset \mathcal{R}(L_\lambda - \lambda I)$. Indeed, $\Phi_k \otimes (b - S(b)) \in \mathcal{R}(A - \lambda I)$ since $P_M(\Phi_k \otimes (b - S(b))) = 0$ and from Lemma 2 we conclude that $T(b) \in \mathcal{R}(L_\lambda - \lambda I)$ for all $b \in E$.

Proof of Lemma 6. We work by induction in m. Consider the operator

$$A^{(2)} = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

with A_1, A_2 Fredholm operators on E and $B \in \mathscr{L}(E)$.

Elementary arguments of linear algebra allow us to ensure that

$$\mathcal{N}(A^{(2)}) = ([\tilde{e}_1, \ldots, \tilde{e}_p] + \mathcal{N}(A_1)) \times (\mathcal{N}(A_2) \cap B^{-1}(\mathcal{R}(A_1))),$$

where $[\tilde{e}_1, \ldots, \tilde{e}_p]$ is the linear subspace generated by the elements $\tilde{e}_j$ such that $A_1\tilde{e}_j = Be_j$, $j = 1, \ldots, p$, and $\{e_1, \ldots, e_p\}$ is one basis of $\mathcal{N}(A_2) \cap B^{-1}(\mathcal{R}(A_1))$. Therefore $\mathcal{N}(A^{(2)})$ is a finite-dimensional subspace of E^2.

Also,

$$\mathcal{R}(A^{(2)}) = \{(v_1, v_2) \in E^2;\ v_2 \in \mathcal{R}(A_2);\ v_1 - B\tilde{A}_2^{-1}v_2 \in \mathcal{R}(A_1) + B(\mathcal{N}(A_1))\},$$

where $\tilde{A}_2$ is an isomorphism obtained by restriction of A_2 to a complementary subspace of $\mathcal{N}(A_2)$.

There exist two finite subsets of E^*, $\{x_1^*, \ldots, x_q^*\}$ and $\{y_1^*, \ldots, y_s^*\}$, such that

(a) $v_2 \in \mathcal{R}(A_2)$ if and only if $\langle x_j^*, v_2\rangle = 0, j = 1, \ldots, q$.

(b) $v_1 - B\tilde{A}_2^{-1} \in \mathcal{R}(A_1) + B(\mathcal{N}(A_2))$ if and only if $\langle y_j^*, v_1 - B\tilde{A}_2^{-1}v_2\rangle = 0, j = 1, \ldots, s$.

From this we conclude that $\mathcal{R}(A^{(2)}) = \mathcal{N}(\mathscr{A})$, where $\mathscr{A}$: $E^2 \to \mathbf{R}^{q+s}$ is defined by

$$\mathscr{A}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \langle x_1^*, v_2\rangle \\ \vdots \\ \langle x_q^*, v_2\rangle \\ \langle y_1^*, v_1 - B\tilde{A}_2^{-1}v_2\rangle \\ \vdots \\ \langle y_s^*, v_1 - B\tilde{A}_2^{-1}v_2\rangle \end{bmatrix}.$$

It is easy to see that $\dim \mathcal{N}(\mathscr{A}) < +\infty$ and then $\mathcal{R}(A^{(2)})$ is a closed subspace of finite codimension in E^2.

With this we have finished the proof that $A^{(2)}$ is a Fredholm operator.

If we accept that $A^{(m-1)}$ is of Fredholm, from the above arguments it follows immediately that $A^{(m)}$ is so as well.

ACKNOWLEDGMENT

The authors are pleased to acknowledge valuable discussions with Professor J. Gutiérrez, notably in relation to the definition of L^*.

REFERENCES

1. O. ARINO, Some spectral properties for the asymptotic behavior of semigroups connected to population dynamics, *SIAM Rev.* **34**, No. 4 (1992), 445–476.
2. O. ARINO AND M. KIMMEL, Asymptotic analysis of a cell cycle model based on unequal division, *SIAM J. Appl. Math.* **47** (1987), 128–145.
3. O. ARINO AND M. KIMMEL, Asymptotic behavior of a nonlinear functional-integral equation of cell kinetics with unequal division, *J. Math. Biol.* **27** (1989), 341–354.
4. O. ARINO AND A. MORTABIT, Slow oscillations in a model of cell population dynamics, *in* "Lecture Notes in Pure and Applied Mathematics, Vol. 131," Dekker, New York, 1991.
5. C. BERNIER AND A. MANITIUS, On semigroups in $\mathbf{R}^n \times L^p$ corresponding to differential equations with delay, *Canad. J. Math.* **30** (1978), 897–914.
6. J. DIESTEL AND J. J. UHL, "Vector Measures," Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.
7. N. DUNFORD AND J. SCHWARTZ, "Linear Operators," Part I, "General Theory," Interscience, New York, 1958.
8. A. GRABOSCH AND U. MOUSTAKAS, A semigroup approach to retarded differential equations, *in* "One-Parameter Semigroups of Positive Operators (R. Nagel, ed.), pp. 219–232, Lecture Notes in Mathematics, Vol. 1184, Springer-Verlag, Berlin/New York, 1986.
9. J. HALE, "Functional Differential Equations," Springer-Verlag, Berlin/New York, 1977.
10. T. KATO, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin/New York, 1966.
11. M. KIMMEL, Z. DARZYNKIEWICZ, O. ARINO, AND F. TRAGANOS, Analysis of a cell cycle model based on unequal division of metabolic constituents to daughter cells during cytokinesis, *J. Theoret. Biol.* **110** (1984), 637–664.
12. K. KUNISH AND W. SCHAPPACHER, Variation of constants formula for partial differential equations with delay, *Nonlinear Anal. TMA* **5** (1981), 123–142.
13. K. KUNISH AND W. SCHAPPACHER, Necessary conditions for partial differential equations with delay to generate c_0-semigroups, *J. Differential Equations* **50** (1983), 49–79.
14. A. MANITIUS, Completeness and F-completeness of eigenfunctions associated with retarded functional differential equations. *J. Differential Equations* **35** (1980), 1–29.
15. S. NAKAGIRI, On the fundamental solution of delay-differential equations in Banach spaces, *J. Differential Equations* **41** (1981), 349–368.
16. S. NAKAGIRI, Pointwise completeness and degeneracy of functional differential equations in Banach spaces. I. General time delays, *J. Math. Anal. Appl.* **127** (1987), 492–529.
17. S. NAKAGIRI, Structural properties of functional differential equations in Banach spaces, *Osaka J. Math.* **25** (1988), 353–398.
18. W. RUDIN, "Functional Analysis," McGraw–Hill, New York, 1973.
19. E. SANCHEZ, O. ARINO, AND M. KIMMEL, Noncompact semigroups of operators generated by cell kinetics models, *Differential Integral Equations* **4**, No. 6 (1991), 1233–1249.

20. H. H. SCHAEFER, "Topological Vector Spaces," Springer-Verlag, Berlin/New York, 1971.
21. K. SHUMACHER, On the resolvent of linear nonautonomous partial differential equations, *J. Differential Equations* **59** (1985), 355–387.
22. A. TAYLOR, "Introduction to Functional Analysis," Wiley, New York, 1967.
23. C. C. TRAVIS AND G. F. WEBB, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974), 395–418.
24. G. F. WEBB, "Theory of Nonlinear Age-Dependent Population Dynamics, Dekker, New York, 1985.