Journal of Mathematical Analysis and Applications **235,** 435–453 (1999)
Article ID jmaa.1999.6365, available online at http://www.idealibrary.com on IDEAL®

# An Abstract Neutral Functional Differential Equation Arising From a Cell Population Model

O. Arino

*Laboratoire de Mathématiques Appliquées, I.P.R.A., Av. de l'Université, 64000 Pau, France*

and

O. Sidki

*Faculté des Sciences et Techniques Fes-Saïss, B.P. 2202, Fes, Maroc*

*Submitted by G. F. Webb*

Received July 9, 1998

In this article, an abstract (infinite dimensional) neutral functional differential equation arising from a cell population model is exhibited. It is then shown that a large class of such equations can be solved by means of the theory of nonlinear semigroups. Finally, application to the model equation is detailed. 

## 1. INTRODUCTION

In this article, we consider the model of cell proliferation described by the equation

$$n(t, x) = 2H[N(t)] \int_0^{+\infty} \int_0^{+\infty} f(x, \phi(\tau, \xi)) \gamma(\tau, \xi) n(t - \tau, \xi)\, d\xi\, d\tau, \tag{1}$$

in which

$$N(t) = \int_0^{+\infty} \int_0^{+\infty} \int_{t-\tau}^{t} n(s, y) \gamma(\tau, y)\, ds\, d\tau\, dy.$$



0022-247X/99 $30.00

This model is a modification of a model first proposed in Kimmel *et al.* [9]. The original model was linear.

Nonlinear variants were later considered in Arino and Kimmel [1, 2] and Arino and Mortabit [4]. The model is based on the subdivision of the cell cycle into four consecutive phases represented as follows:

$$\to y \boxed{G_1 \mid S \mid G_2 \mid M} \to y \sim f(\cdot, x) \to$$

$$\tau \sim \gamma(., y) \qquad x = \Phi(\tau, y).$$

During its progression inside the cycle, a cell keeps growing, although with a variable strength. Generally, it doubles its size from birth (as a daughter cell) to the end of the $G_2$ phase before the M phase (mitosis) were it divides into two identical cells, each with half the constituents of the mother cell.

The main hypothesis introduced in [9], in the case of subdivision of the cell, is that the division is not equal.

Equation (1) takes into consideration two further aspects, compared to the one aspect in [9].

One assumes here that the life duration of a cell (that is, the length of a cycle) is not determined by the initial size, but dependence is, in probability, determined by a conditional density $\gamma(., \xi)$ (conditioned on the size): $\int_{\tau_1}^{\tau_2} \gamma(\tau, \xi)\, d\tau =$ Probability for the lifelength of a cell, with initial size $\xi$, to lie within the interval $[\tau_1, \tau_2]$.

As a result, the final size of cells cannot be expressed in terms of their initial size only (as was assumed in [9]) but it is also a function of the lifelength: $\phi(\tau, \xi)$.

A linear model based on these considerations was presented in [3]. Here, we introduce the limiting effects due to the environment in the form of a function of the total population, which decays to zero when population grows to $+\infty$. Such models were considered in [1, 2]. Equation (1) collects and extends two previous models: a linear by Arino, *et al.* [3] and a nonlinear by Arino and Kimmel [1]. Equation (1) is an integral equation and was studied as such in [1, 3].

Here we transform the integral equation into a functional differential equation of neutral type (NFDE) that we solve under suitable hypotheses. Thus, we obtain both a strict extension of previous existence results [1, 3] and novel regularity properties verified by the solutions of the NFDE.

Differentiating (formally) Eq. (1) with respect to time yields the NFDE,

$$\frac{\partial}{\partial t} n(t, .) = G(n_t)(.), \tag{2}$$

where $G$ is a nonlinear operator defined from $W^{1,1}([-r,0]; L^1(0,+\infty))$ into $L^1(0,+\infty)$ by

$$\begin{aligned} G(\varphi) &= 2K(\varphi)\dot{H}[\mathscr{L}(\varphi)] \\ &\quad \times \int_0^{+\infty}\int_0^{+\infty} f(.,\phi(\tau,\xi))\gamma(\tau,\xi)\varphi(-\tau,\xi)\,d\xi\,d\tau \\ &\quad + 2H[\mathscr{L}(\varphi)] \\ &\qquad \times \int_0^{+\infty}\int_0^{+\infty} f(.,\phi(\tau,\xi))\gamma(\tau,\xi)\frac{\partial}{\partial\theta}\varphi(-\tau,\xi)\,d\xi\,d\tau \end{aligned} \tag{3}$$

with

$$\begin{cases} K(\varphi) = \displaystyle\int_0^{+\infty}\int_0^{+\infty}\int_{-\tau}^{0}\frac{\partial}{\partial\theta}\varphi(\theta,\xi)\gamma(\tau,y)\,d\theta\,d\tau\,dy \\ \mathscr{L}(\varphi) = \displaystyle\int_0^{+\infty}\int_0^{+\infty}\int_{-\tau}^{0}\varphi(\theta,\xi)\gamma(\tau,y)\,d\theta\,d\tau\,dy \end{cases},$$

for every $\varphi \in W^{1,1}([-r,0; L^1(0,+\infty))$.

Under appropriate assumptions on the parameters defining Eq. (1), one shows that, for each initial value $n_0 \in W^{1,1}([-r,0]; L^1(0,+\infty))$ and each $T > 0$, NFDE (2) possesses one and only one solution $n$, $n \in W^{1,1}([-r,T]; L^1(0,+\infty))$.

Integrating (2), one obtains the solution of (1) and, in the same way as in [2], one shows that it is nonnegative for all $t \geq 0$, if $n_0 \geq 0$.

## 2. RESOLUTION OF THE NFDE: $\dot{x}(t) = F(x_t)$

In order to study Eq. (2), we give some results on the following class of NFDEs.

$$\frac{dx}{dt} = F(x_t), \qquad x_0 = \varphi \in W^{1,1}([-r,0]; X), \qquad 0 \leq t \leq T, \tag{4}$$

where $x: [-r,T] \to X$, $0 < r < +\infty$ is the delay, $X$ is a Banach space with norm $|.|_X$, and $x_t$ is the history defined pointwise by $x_t(\theta) = x(t+\theta)$, for all $\theta \in [-r,0]$.

We suppose that $F: W^{1,1}([-r,0]; X) \to X$, is Lipschitz continuous, with Lipschitz constant $\alpha$; i.e.,

$$H_{(F)}: |F(\varphi_1) - F(\varphi_2)|_X \leq \alpha\|\varphi_1 - \varphi_2\|_{1,1},$$

$$\text{for all } \varphi_1, \varphi_2 \in W^{1,1}([-r,0]; X).$$

And, we define the space $W^{1,1}([a,b];X)$ by

$$W^{1,1}([a,b];X) = \left\{ \begin{array}{l} f \in L^1([a,b];X);\ f \text{ absolutely continuous on } [a,b], \\ f' \text{ exists a.e., } f' \in L^1([a,b];X), \\ \text{and } f(t) = f(a) + \int_a^t f'(s)\,ds, \qquad \forall t \in [a,b] \end{array} \right\}.$$

For all $f \in W^{1,1}([a,b];X)$, we define the following norm:

$$\|f\|_{1,1} = \int_a^b |f(s)|\,ds + \int_a^b |f'(s)|\,ds. \tag{5}$$

Note by [5] that if $\dim X < +\infty$, or $X$ is a reflexive Banach space, then each absolutely continuous function $x$: $[a,b] \to X$, is a.e differentiable, and $x(t) = x(a) + \int_a^b x'(s)\,ds$.

In [7], Dyson and Villella-Bressan proved the existence and uniqueness of solutions of NFDE (4), written in the form of an evolution equation

$$\frac{du}{dt} = Au,$$

where $A$ is a nonlinear operator defined on $W^{2,1}([-r,0];X)$ with values in $W^{1,1}([-r,0];X)$ by

$$A\varphi = \dot{\varphi} \quad \text{and} \quad \varphi \in D(A) = \{\varphi \in W^{2,1}([-r,0];X) : \dot{\varphi}(0) = F(\varphi)\}. \tag{6}$$

These authors proved that if $F$ satisfies $H_{(F)}$, then $A$ is $(1+\alpha)$-dissipative and such that $\mathrm{Im}(I - \lambda A) = W^{1,1}([-r,0];X)$, for $\lambda > 0$ small enough. Therefore, by the Crandall and Liggett theorem [6], $\lim_{n \to +\infty}(I - (t/n)A)^{-n}\varphi$ exists for all $t \geq 0$ and for all $\varphi \in W^{1,1}([-r,0];X)$. If we define $T(t)\varphi = \lim_{n \to +\infty}(I - \frac{t}{n}A)^{-n}\varphi$, then $T(t)$ is a nonlinear strongly continuous semigroup of type $(1+\alpha)$ on $W^{1,1}([-r,0];X)$. More precisely, it is proved in [7] that:

THEOREM 1 [7]. *Let $F$ satisfy $H_{(F)}$. Then, $A$ defined by* (6) *generates a semigroup $T(t)$ of type* $(1+\alpha)$ *in* $W^{1,1}([-r,0];X)$. *Set*

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r,0] \\ (T(t)\varphi)(0) & \text{if } t > 0. \end{cases}$$

*Then $x(t)$ is the unique strong solution of Eq.* (4) *for all* $\varphi \in W^{1,1}([-r,0];X)$. *If* $\varphi \in E = \{\varphi \in \mathscr{C}^1([-r,0];X) : \dot{\varphi}(0) = F(\varphi)\}$, *then the solution is continuously differentiable.*

In [10], we proposed another method. A direct approach by means of an integral equation was considered. More precisely, we proved that there exists an integer $N$, such that $K^n$ is a strict contraction, for all $n \geq N$, where $K$ is given by

$$(Kx)(t) = \begin{cases} \varphi(0) + \int_0^t F(x_s)\, ds & \text{if } t > 0 \\ \varphi(t) & \text{if } t \in [-r,0] \end{cases}.$$

$K$ is defined on the set $E_\varphi = \{y \in W^{1,1}([-r,T];X) : y = \varphi \text{ on } [-r,0]\}$, where $T > 0$ is arbitrary and $\varphi \in W^{1,1}([-r,0];X)$ is the initial datum of (4).

In [10] the following theorem was proved:

THEOREM 2. *Let $F$ satisfy $H_{(F)}$ and $E_\varphi = \{y \in W^{1,1}([-r,T];X) : y = \varphi$ on $[-r,0]\}$. Then, NFDE* (4) *has a unique solution $x \in E_\varphi$, for all $T > 0$, and for each initial data $\varphi \in W^{1,1}([-r,0];X)$.*

Set

$$T(t)\varphi = x_t, \tag{7}$$

with

$$x_t(\theta) = \begin{cases} x(t+\theta) & \text{if } t+\theta > 0 \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0 \end{cases}. \tag{8}$$

Finally, we quote the following proposition that will be used for the resolution of NFDE (2).

PROPOSITION 3 [10]. *Suppose that $F$ verifies $H_{(F)}$. Then,* (a) *The family of operators $\{T(t)\}_{t\geq 0}$ defined from $W^{1,1}([-r,0];X)$ into $W^{1,1}([-r,0];X)$ by* (7) *is a nonlinear strongly continuous semigroup that verifies the following relation: for all $\theta \in [-r,0]$,*

$$(T(t))\varphi(\theta) = \begin{cases} \varphi(0) + \int_0^{t+\theta} F(T(s)\varphi)\, ds & \text{if } t+\theta > 0 \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0 \end{cases}. \tag{9}$$

(b) *For all $\varphi_1, \varphi_2 \in W^{1,1}([-r,0];X)$ and all $t \geq 0$, we have*

$$\|T(t)\varphi_1 - T(t)\varphi_2\|_{1,1} \leq e^{(\alpha+1)t}\|\varphi_1 - \varphi_2\|_{1,1}. \tag{10}$$

(c) *The operator $A$ defined by* (6) *is the infinitesimal generator of $T(t)$.*

## 3. RESOLUTION OF THE NFDE: $(\partial/\partial t)n(t,.) = G(n_t)(.)$

The parameters of Eq. (1) are the functions $f$, $\phi$, $\gamma$, and $H$. We make the following hypotheses:

$H_{(f)}$: $f \in L^1(IR_+^2)$, $f \geq 0$, and $\int_0^{+\infty} f(y,x)\,dy = 1$, for all $x \geq 0$.

$H_{(\gamma)}$: $\gamma \in L^1(IR_+^2)$, $\gamma \geq 0$, $\int_0^{+\infty} \gamma(\tau,x)\,d\tau = 1$ and there exists a constant $k > 0$, such that $|\gamma(\tau,x)| \leq k$, for all $(\tau,x) \in [0,+\infty) \times [0,+\infty)$. There exist $A_1$, $A_2$, $\tau_1$, and $\tau_2$ such that $0 < \tau_1 < \tau_2$, $0 < A_1 < A_2$, and $\operatorname{supp} \gamma(.,\xi) \subset [\tau_1, \tau_2]$, for all $\xi \in [A_1, A_2]$.

$H_{(\phi)}$: $\phi \in \mathscr{C}(IR_+^2)$ and $\phi \geq 0$.

$H_{(H)}$: $H \in \mathscr{C}^1(IR)$, $0 \leq H \leq 1$, and $H, \dot{H}$ are locally Lipschitz.

We define the norm in $W^{1,1}([-r,0]; L^1(0,+\infty))$ by

$$\|\varphi\|_{1,1} = \int_{-r}^{0} |\varphi(\theta,.)|_{L^1}\,d\theta + \int_{-r}^{0} |\dot{\varphi}(\theta,.)|_{L^1}\,d\theta \tag{11}$$

for all $\varphi \in W^{1,1}([-r,0]; L^1(0,+\infty))$.

PROPOSITION 4. *We suppose $H_{(f)}$, $H_{(\gamma)}$, $H_{(\phi)}$, and $H_{(H)}$. Then, the map $\varphi \to G(\varphi)(.)\chi_R(\|\varphi\|_{1,1})$ is Lipschitz continuous from $W^{1,1}([-r,0]; L^1(0,+\infty))$ into $L^1(0,+\infty)$, where*

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| \geq 2R \\ -\dfrac{1}{R}x + 2 & \text{if } R \leq x \leq 2R \\ \dfrac{1}{R}x + 2 & \text{if } -2R \leq x \leq -R \end{cases}. \tag{12}$$

*Proof.* Let $\varphi_1, \varphi_2 \in W^{1,1}([-r,0]; L^1(0,+\infty))$. For each $x \geq 0$, we will evaluate the following expression:

$$\begin{aligned} G(\varphi_1)(x)\,\chi_R(\|\varphi_1\|_{1,1}) &- G(\varphi_2)(x)\,\chi_R(\|\varphi_2\|_{1,1}) \\ &= \left[G(\varphi_1)(x) - G(\varphi_2)(x)\right] \chi_R(\|\varphi_2\|_{1,1}) \\ &\quad - G(\varphi_1)(x)\left[\chi_R(\|\varphi_2\|_{1,1}) - \chi_R(\|\varphi_1\|_{1,1})\right]. \end{aligned} \tag{13}$$

We consider three possible cases: ($\|\varphi_1\|_{1,1} \leq R$ and $\|\varphi_2\|_{1,1} \leq R$), ($\|\varphi_1\|_{1,1} \geq 2R$ and $\|\varphi_2\|_{1,1} \geq 2R$), and ($\|\varphi_1\|_{1,1} \leq R$ and $\|\varphi_2\|_{1,1} \geq 2R$).

*Case I.* $\|\varphi_1\|_{1,1} \geq 2R$ and $\|\varphi_2\|_{1,1} \geq 2R$. So, we have

$$G(\varphi_1)(x)\chi_R(\|\varphi_1\|_{1,1}) - G(\varphi_2)(x)\chi_R(\|\varphi_2\|_{1,1}) = 0.$$

*Case II.* $\|\varphi_1\|_{1,1} \leq R$ and $\|\varphi_2\|_{1,1} \geq 2R$.
From (13) one obtains

$$\begin{aligned}
&\left|G(\varphi_1)(x)\chi_R(\|\varphi_1\|_{1,1}) - G(\varphi_2)(x)\chi_R(\|\varphi_2\|_{1,1})\right| \\
&\quad = \left|G(\varphi_1)(x)\right|\left|\chi_R(\|\varphi_2\|_{1,1}) - \chi_R(\|\varphi_1\|_{1,1})\right| \\
&\quad \leq \left|G(\varphi_1)(x)\right| |\chi_R|_{\mathrm{Lip}} \left|\|\varphi_2\|_{1,1} - \|\varphi_1\|_{1,1}\right| \\
&\quad \leq \left|G(\varphi_1)(x)\right| |\chi_R|_{\mathrm{Lip}} \|\varphi_2 - \varphi_1\|_{1,1}.
\end{aligned}$$

So,

$$\begin{aligned}
&\int_0^{+\infty} \left|G(\varphi_1)(x)\chi_R(\|\varphi_1\|_{1,1}) - G(\varphi_2)(x)\chi_R(\|\varphi_2\|_{1,1})\right| dx \\
&\qquad \leq |\chi_R|_{\mathrm{Lip}} \|\varphi_2 - \varphi_1\|_{1,1} \int_0^{+\infty} \left|G(\varphi_1)(x)\right| dx.
\end{aligned}$$

Now, we prove that there exists a constant $C > 0$, such that

$$\int_0^{+\infty} \left|G(\varphi_1)(x)\right| dx \leq C,$$

for all $\varphi_1$, such that $\|\varphi_1\|_{1,1} \leq R$: $\varphi_1$ is defined on $[-r, 0]$, so, in (3) $\tau \in [0, r]$. By a change of variable $\tau' = -\tau$ and integrating (3) from 0 to $+\infty$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \left|G(\varphi)(x)\right| dx \\
&\quad \leq 2\left|K(\varphi)\right|\left|\dot{H}(\mathscr{L}(\varphi))\right| \\
&\qquad \times \left\{\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^{0} \left|f(x, \phi(-\tau, \xi))\right|\left|\gamma(-\tau, \xi)\right|\left|\varphi(\tau, \xi)\right| d\xi\, d\tau\, dx\right\} \\
&\qquad + 2\left|H(\mathscr{L}(\varphi))\right|\left\{\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^{0} \left|f(x, \phi(-\tau, \xi))\right|\left|\gamma(-\tau, \xi)\right| \right. \\
&\qquad\qquad \left. \times \left|\dot{\varphi}(\tau, \xi)\right| d\xi\, d\tau\, dx\right\}.
\end{aligned}$$

From $H_{(\gamma)}$, we have

$$|K(\varphi)| \leq \int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |\dot{\varphi}(\theta, y)||\gamma(\tau, y)|\, d\theta\, d\tau\, dy$$

$$\leq \int_0^{+\infty}\int_{-r}^0 |\dot{\varphi}(\theta, y)|\, d\theta\, dy \leq R \tag{14}$$

and

$$|\mathscr{L}(\varphi)| \leq \int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |\varphi(\theta, y)||\gamma(\tau, y)|\, d\theta\, d\tau\, dy$$

$$\leq \int_0^{+\infty}\int_{-r}^0 |\varphi(\theta, y)|\, d\theta\, dy \leq R. \tag{15}$$

Then, there exists a constant $k_1 > 0$, such that

$$\left|\dot{H}(\mathscr{L}(\varphi))\right| \leq k_1. \tag{16}$$

So, from $H_{(f)}$, we obtain

$$\int_0^{+\infty} |G(\varphi)(x)|\, dx$$
$$\leq 2k_1 R \int_0^{+\infty}\int_{-r}^0 |\varphi(\tau, \xi)|\, d\xi\, d\tau + 2\int_0^{+\infty}\int_{-r}^0 |\dot{\varphi}(\tau, \xi)|\, d\xi\, d\tau$$
$$\leq 2(k_1 R + 1) R.$$

Finally, we have

$$\left|G(\varphi_1)(x)\,\chi_R\left(\|\varphi_1\|_{1,1}\right) - G(\varphi_2)(x)\,\chi_R\left(\|\varphi_2\|_{1,1}\right)\right|_{L^1} \leq \lambda_1 \|\varphi_1 - \varphi_2\|_{1,1} \tag{17}$$

with $\lambda_1 = 2(k_1 R + 1)R|\chi_R|_{\mathrm{Lip}}$.

*Case III*. $\|\varphi_1\|_{1,1} \leq R$ and $\|\varphi_2\|_{1,1} \leq R$. Therefore,

$$G(\varphi_1)(x)\,\chi_R\left(\|\varphi_1\|_{1,1}\right) - G(\varphi_2)(x)\,\chi_R\left(\|\varphi_2\|_{1,1}\right)$$
$$= G(\varphi_1)(x) - G(\varphi_2)(x)$$
$$= 2K(\varphi_1)\dot{H}(\mathscr{L}(\varphi_1)) \int_0^{+\infty}\int_{-r}^0 f(x, \phi(-\tau, \xi))$$
$$\times \gamma(-\tau, \xi)\,\varphi_1(\tau, \xi)\, d\xi\, d\tau$$
$$+ 2H\left(\mathscr{L}(\varphi_1)\right) \int_0^{+\infty}\int_{-r}^0 f(x, \phi(-\tau, \xi))$$

$$\times \gamma(-\tau,\xi)\dot{\varphi}_1(\tau,\xi)\,d\xi\,d\tau$$

$$- 2K(\varphi_2)\dot{H}(\mathscr{L}(\varphi_2))\int_0^{+\infty}\int_{-r}^0 f(x,\phi(-\tau,\xi))$$

$$\times \gamma(-\tau,\xi)|\varphi_2(\tau,\xi)|\,d\xi\,d\tau$$

$$- 2H(\mathscr{L}(\varphi_2))\int_0^{+\infty}\int_{-r}^0 f(x,\phi(-\tau,\xi))$$

$$\times \gamma(-\tau,\xi)\dot{\varphi}_2(\tau,\xi)\,d\xi\,d\tau.$$

So, $\int_0^{+\infty}|G(\varphi_1)(x) - G(\varphi_2)(x)|dx \le I_1 + I_2 + I_3 + I_4$, where

$$I_1 = 2|K(\varphi_1)||\dot{H}(\mathscr{L}(\varphi_1))|$$

$$\times\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |f(x,\phi(-\tau,\xi))||\gamma(-\tau,\xi)|$$

$$\times|\varphi_1(\tau,\xi) - \varphi_2(\tau,\xi)|\,d\xi\,d\tau\,dx,$$

$$I_2 = 2|K(\varphi_1)\dot{H}(\mathscr{L}(\varphi_1)) - K(\varphi_2)\dot{H}(\mathscr{L}(\varphi_2))|$$

$$\times\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |f(x,\phi(-\tau,\xi))||\gamma(-\tau,\xi)|$$

$$\times|\varphi_2(\tau,\xi)|\,d\xi\,d\tau\,dx,$$

$$I_3 = 2|H(\mathscr{L}(\varphi_2))|$$

$$\times\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |f(x,\phi(-\tau,\xi))||\gamma(-\tau,\xi)|$$

$$\times|\dot{\varphi}_2(\tau,\xi) - \dot{\varphi}_1(\tau,\xi)|\,d\xi\,d\tau\,dx,$$

and,

$$I_4 = 2|H(\mathscr{L}(\varphi_1)) - H(\mathscr{L}(\varphi_2))|$$

$$\times\int_0^{+\infty}\int_0^{+\infty}\int_{-r}^0 |f(x,\phi(-\tau,\xi))||\gamma(-\tau,\xi)||\dot{\varphi}_2(\tau,\xi)|\,d\xi\,d\tau\,dx.$$

In view of $H_{(f)}$, $H_{(\gamma)}$, $H_{(H)}$, (14), (15), and (16), we have

$$I_1 \le 2Rkk_1\int_0^{+\infty}\int_{-r}^0 |\varphi_1(\tau,\xi) - \varphi_2(\tau,\xi)|\,d\xi\,d\tau \le \lambda_2\|\varphi_1 - \varphi_2\|_{1,1}$$

with $\lambda_2 = 2Rkk_1$,

$$\begin{aligned} I_2 &\le 2Rk\Big\{\big|K(\varphi_1) - K(\varphi_2)\big|\big|\dot{H}(\Lambda(\varphi_1))\big| \\ &\qquad + \big|K(\varphi_2)\big|\big|\dot{H}(\mathscr{L}(\varphi_1)) - \dot{H}(\mathscr{L}(\varphi_2))\big|\Big\} \\ &\le 2Rkk_1\int_0^{+\infty}\int_{-r}^0 \big|\dot{\varphi}_1(\tau,\xi) - \dot{\varphi}_2(\tau,\xi)\big|\,d\xi\,d\tau \\ &\qquad + 2R^2k|\dot{H}|_{\text{Lip}}\int_0^{+\infty}\int_{-r}^0 \big|\varphi_1(\tau,\xi) - \varphi_2(\tau,\xi)\big|\,d\xi\,d\tau \\ &\le \lambda_3\|\varphi_1 - \varphi_2\|_{1,1} \end{aligned}$$

with $\lambda_3 = 2Rk(k_1 + R)$,

$$I_3 \le 2k\int_0^{+\infty}\int_{-r}^0 \big|\dot{\varphi}_1(\tau,\xi) - \dot{\varphi}_2(\tau,\xi)\big|\,d\xi\,d\tau \le \lambda_4\|\varphi_1 - \varphi_2\|_{1,1}$$

with $\lambda_4 = 2k$, and

$$I_4 \le 2|H|_{\text{Lip}}\,k^2R\int_0^{+\infty}\int_{-r}^0 \big|\varphi_1(\tau,\xi) - \varphi_2(\tau,\xi)\big|\,d\xi\,d\tau \le \lambda_5\|\varphi_1 - \varphi_2\|_{1,1},$$

with $\lambda_5 = 2|H|_{\text{Lip}}\,k^2R$.

So,

$$\begin{aligned} \int_0^{+\infty}\big|G(\varphi_1)(x) - G(\varphi_2)(x)\big|\,dx &\le I_1 + I_2 + I_3 + I_4 \\ &\le (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)\|\varphi_1 - \varphi_2\|_{1,1}. \end{aligned}$$

Then, there exists a constant $\alpha > 0$, such that

$$\big|G(\varphi_1)(.)\,\chi_R(\|\varphi_1\|_{1,1}) - G(\varphi_2)(.)\,\chi_R(\|\varphi_2\|_{1,1})\big|_{L^1} \le \alpha\|\varphi_1 - \varphi_2\|_{1,1}.$$

∎

Thus, by Theorem 2, the problem

$$\begin{cases} \dfrac{\partial}{\partial t}n(t,.) = G(n_t)(.)\,\chi_R(\|n_t\|_{1,1}) \\ n_0 = \varphi \end{cases} \tag{18}$$

has a unique solution $n \in W^{1,1}([-r, T]; L^1(0, +\infty))$, for all $T > 0$, and for all initial data $\varphi \in W^{1,1}([-r, 0]; L^1(0, +\infty))$.

## 4. APPLICATION TO THE INTEGRAL EQUATION

For all $R > 0$ and for all $\varphi \in W^{1,1}([-r, 0]; L^1(0, +\infty))$, we define

$$T_R(t)\varphi = n_t. \tag{19}$$

Then, by Proposition 3, the family $\{T_R(t)\}_{t \geq 0}$ is a nonlinear strongly continuous semigroup of type $(\alpha + 1)$ on $W^{1,1}([-r, 0]; L^1(0, +\infty))$, and verifies the relation

$$(T_R(t)\varphi)(\theta, x) = \begin{cases} \varphi(0, x) + \int_0^{t+\theta} G(T_R(s)\varphi)(x)\,\chi_R(\|T_R(s)\varphi\|_{1,1})\, ds \\ \qquad\qquad \text{if } t + \theta > 0 \\ \varphi(t + \theta, x) \quad \text{if } -r \leq t + \theta \leq 0, \end{cases} \tag{20}$$

and the infinitesimal generator of $T_R(t)$ is given by

$$\begin{aligned} A_R\varphi &= \dot{\varphi} \\ D(A_R) &= \{\varphi \in W^{2,1}([-r, 0]; L^1(0, +\infty)) : \\ &\qquad \dot{\varphi}(0, .) = G(\varphi)(.)\,\chi_R(\|\varphi\|_{1,1})\}. \end{aligned} \tag{21}$$

The map $t \mapsto T_R(t)\varphi$ is continuous on $[0, +\infty)$. So, for all $\varphi$ in the ball $B(0, R)$, there exists $t(\varphi, R) > 0$ such that $\|T_R(t)\varphi\|_{1,1} < R$, for all $t < t(\varphi, R)$. Then we set

$$\tau_R(\varphi) = \sup\{s > 0, \|T_R(t)\varphi\|_{1,1} < R, 0 \leq t \leq s\}. \tag{22}$$

Note that the map $R \mapsto \tau_R(\varphi)$ is monotone strictly increasing on $]0, +\infty[$.

We will now prove that essentially $T_R(t)\varphi$ is independent on $R$, for all $\varphi \in B(0, R)$, provided that $t < t(\varphi, R)$. In fact,

PROPOSITION 5. *For all $R_1, R_2 > 0$, such that $R_2 > R_1$ and for all $\varphi \in B(0, R_1)$, we have $T_{R_1}(t)\varphi = T_{R_2}(t)\varphi$, for all $t \in [0, \tau_{R_1}(\varphi)[$.*

*Proof.* Let $t \in [0, \tau_{R_1}(\varphi)[$. So, $t < \tau_{R_1}(\varphi) < \tau_{R_2}(\varphi)$, therefore

$$\chi_{R_1}\left(\|T_{R_1}(t)\varphi\|_{1,1}\right) = \chi_{R_2}\left(\|T_{R_2(t)\varphi}\|_{1,1}\right) = 1$$

and

$$\begin{aligned}
&\left\| T_{R_1}(t)\varphi - T_{R_2}(t)\varphi \right\|_{1,1} \\
&\quad = \int_0^{+\infty}\int_{-r}^0 \left| T_{R_1}(t)\varphi(\theta,x) - T_{R_2}(t)\varphi(\theta,x)\right| d\theta\, dx \\
&\qquad + \int_0^{+\infty}\int_{-r}^0 \left| \frac{\partial}{\partial\theta} T_{R_1}(t)\varphi(\theta,x) - \frac{\partial}{\partial\theta} T_{R_2}(t)\varphi(\theta,x)\right| d\theta\, dx \\
&\quad = \int_0^{+\infty}\int_{-t}^0 \left| T_{R_1}(t+\theta)\varphi(0,x) - T_{R_2}(t+\theta)\varphi(0,x)\right| d\theta\, dx \\
&\qquad + \int_0^{+\infty}\int_{-t}^0 \left| \frac{\partial}{\partial\theta} T_{R_1}(t+\theta)\varphi(0,x) - \frac{\partial}{\partial\theta} T_{R_2}(t+\theta)\varphi(0,x)\right| d\theta\, dx \\
&\quad = \int_0^{+\infty}\int_0^t \left| T_{R_1}(\eta)\varphi(0,x) - T_{R_2}(\eta)\varphi(0,x)\right| d\eta\, dx \\
&\qquad + \int_0^{+\infty}\int_0^t \left| \frac{\partial}{\partial\eta} T_{R_1}(\eta)\varphi(0,x) - \frac{\partial}{\partial\eta} T_{R_2}(\eta)\varphi(0,x)\right| d\eta\, dx \\
&\quad = \int_0^{+\infty}\int_0^t \left| \int_0^\eta \left[ G\big(T_{R_1}(s)\varphi\big)(x) - G\big(T_{R_2}(s)\varphi\big)(x)\right] ds \right| d\eta\, dx \\
&\qquad + \int_0^{+\infty}\int_0^t \left| G\big(T_{R_1}(\eta)\varphi\big)(x) - G\big(T_{R_2}(\eta)\varphi\big)(x)\right| d\eta\, dx \\
&\quad \le t\alpha \int_0^t \left\| T_{R_1}(s)\varphi - T_{R_2}(s)\varphi \right\|_{1,1} ds \\
&\qquad + \alpha \int_0^t \left\| T_{R_1}(\eta)\varphi - T_{R_2}(\eta)\varphi \right\|_{1,1} d\eta \\
&\quad \le \alpha\big(1 + \tau_{R_1}(\varphi)\big) \int_0^t \left\| T_{R_1}(s)\varphi - T_{R_2}(s)\varphi \right\|_{1,1} ds.
\end{aligned}$$

So, by Gronwall's lemma, we have,

$$\left\| T_{R_1}(t)\varphi - T_{R_2}(t)\varphi \right\|_{1,1} = 0, \quad \text{for all } t \in [0, \tau_{R_1}(\varphi)[.$$

∎

From (19), we have $T_R(t)\varphi = n_t$. Thus,

$$n(t,x) = \begin{cases} \big(T_R(t)\varphi\big)(0,x) & \text{if } t > 0 \\ \varphi(t,x) & \text{if } t \in [-r, 0]. \end{cases} \tag{23}$$

Let $R_0$, such that $\tau_{R_0}(\varphi) = \sup_{R > \|\varphi\|_{1,1}} \tau_R(\varphi)$ and $[0, \tau_{R_0}(\varphi)[$ the maximal interval where a solution $n$ of (18) is defined. Then, for each $T < \tau_{R_0}(\varphi)$, $n_{|[0,T]} \in W^{1,1}([0,T]; L^1(0,+\infty))$ and is unique as a solution of (18).

We will now show that $\tau_{R_0}(\varphi) = +\infty$. Indeed, we have the following:

PROPOSITION 6. *If $\tau_{R_0}(\varphi) < +\infty$, then*

$$\limsup_{t \to \tau_{R_0}^-} \|\eta_t\|_{1,1} = +\infty.$$

*Proof.* The proof is done by contradiction. Suppose that there exists $k > 0$, such that $\|n_t\|_{1,1} \le k$, for all $t \in [0, \tau_{R_0}(\varphi)[$. Then, by (19),

$$\left\| T_{R_0}(t)\varphi \right\|_{1,1} \le k, \quad \text{for all } t \in \left[0, \tau_{R_0}(\varphi)\right[.$$

We have $\tau_k(\varphi) \le \tau_{R_0}(\varphi)$, therefore $k \le R_0$. So,

$$\left\| T_{R_0}(t)\varphi \right\|_{1,1} \le R_0 \quad \text{for all } t \in \left[0, \tau_{R_0}(\varphi)\right[.$$

There exists $T > 0$, such that $\tau_{R_0}(\varphi) - T/2 < T < \tau_{R_0}(\varphi)$. So, Eq. (18) has a unique solution on $[0,T]$, for the initial data $\tilde{\varphi} = T_{R_0}(\tau_{R_0}(\varphi) - T/2)\varphi$.

$$\|\tilde{\varphi}\|_{1,1} = \left\| T_{R_0}\left(\tau_{R_0}(\varphi) - \frac{T}{2}\right)\varphi \right\|_{1,1} < R_0 \left(\text{because } \tau_{R_0}(\varphi) - \frac{T}{2} < \tau_{R_0}(\varphi)\right).$$

The solution of Eq. (18) is given by

$$n_t = T_{R_0}(t)\tilde{\varphi} = T_{R_0}(t) T_{R_0}\left(\tau_{R_0}(\varphi) - \frac{T}{2}\right)\varphi = T_{R_0}\left(t + \tau_{R_0}(\varphi) - \frac{T}{2}\right)\varphi \quad \text{for all } t \in [0,T].$$

We denote $\xi = t + \tau_{R_0}(\varphi) - T/2$. Then, $\xi \in [0, \tau_{R_0}(\varphi) + T/2[$ and $\eta_\xi = T_{R_0}(\xi)\varphi$ is a solution of (18) on $[0, \tau_{R_0}(\varphi) + T/2[$, for the initial data $\varphi$. This is a contradiction with the maximality of interval $[0, \tau_{R_0}(\varphi)[$. So,

$$\limsup_{t \to \tau_{R_0}} \|n_t\|_{1,1} = +\infty.$$

∎

From the fact that $\|n_t\|_{1,1} < R_0$, for all $t \in [0, \tau_{R_0}(\varphi)[$, we have $\tau_{R_0}(\varphi) = +\infty$. But, $\tau_{R_0}(\varphi) = \sup_{R > \|\varphi\|_{1,1}} \tau_R(\varphi) = \lim_{R \to +\infty} \tau_R(\varphi) = +\infty$.

Then, we define $T(t)\varphi = T_{R_0}(t)\varphi$ for all $t \in [0, +\infty)$ and all $\varphi \in W^{1,1}$.

So, by (20) we have for all $\varphi \in W^{1,1}$,

$$n_t(\theta, x) = T(t)\varphi(\theta, x)$$
$$= \begin{cases} \varphi(0, x) + \int_0^{t+\theta} G(T(s)\varphi)(x)\, ds & \text{if } t + \theta > 0 \\ \varphi(t + \theta, x) & \text{if } -r \le t + \theta \le 0. \end{cases} \tag{24}$$

The infinitesimal generator of $T(t)$ is given by

$$A\varphi = \dot{\varphi}, \qquad D(A) = \{\varphi \in W^{2,1}, \dot{\varphi}(0, .) = G(\varphi)(.)\}. \tag{25}$$

So,

$$n(t, x) = \begin{cases} (T(t)\varphi)(0, x) & \text{if } t > 0 \\ \varphi(t, x) & \text{if } t \in [-r, 0] \end{cases} \tag{26}$$

is the unique solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(t, .) = G(n_t)(.) \\ n_0 = \varphi \end{cases}. \tag{27}$$

Finally, we have the following theorem:

THEOREM 7. *For all* $\varphi \in W^{1,1}([-r, 0]; L^1(0, +\infty))$ *and all* $T \in [0, +\infty)$, *problem* (27) *has a unique solution* $n \in W^{1,1}([-r, T]; L^1(0, +\infty))$, *given by* (26).

For all $\varphi \in W^{1,1}([-r, 0]; L^1(0, +\infty))$, we define

$$\mathscr{L}(\varphi) = \int_0^{+\infty} \int_0^{+\infty} \int_{-\tau}^{0} \varphi(\theta, y)\gamma(\tau, y)\, d\tau\, dy$$

and

$$\mathscr{A}(\varphi)(x) = \int_0^{+\infty} \int_0^{+\infty} f(x, \phi(\tau, \xi))\gamma(\tau, \xi)\varphi(-\tau, \xi)\, d\xi\, d\tau.$$

So, $\mathscr{L}(n_t) = N(t)$ and $\mathscr{A}(n_t)(x) = \int_0^{+\infty}\int_0^{+\infty} f(x, \phi(\tau, \xi))\gamma(\tau, \xi)n(t - \tau, \xi)\, d\xi\, d\tau$. Thus, Eq. (1) becomes

$$n(t, x) = 2H(\mathscr{L}(n_t))\mathscr{A}(n_t)(x). \tag{28}$$

In order to return to Eq. (28), we integrate (27) from 0 to $t$, for all $x \geq 0$:

$$\int_0^t \frac{\partial}{\partial s} n(s, x)\, ds = \int_0^t 2\frac{\partial}{\partial s} H(\mathscr{L}(n_s))\mathscr{A}(n_s)(x)\, ds.$$

It is equivalent to

$$n(t, x) - n(0, x) = 2H(\mathscr{L}(n_t))\mathscr{A}(n_t)(x) - 2H(\mathscr{L}(n_0))\mathscr{A}(n_0)(x),$$

i.e.,

$$n(t, x) = 2H(\mathscr{L}(n_t))\mathscr{A}(n_t)(x) + \left[n(0, x) - 2H(\mathscr{L}(n_0))\mathscr{A}(n_0)(x)\right].$$

So, an additional condition comes out:

$$n_0(0, .) = 2H(\mathscr{L}(n_0))\mathscr{A}(n_0)(.). \tag{29}$$

PROPOSITION 8. *Suppose $H_{(f)}$, $H_{(\gamma)}$, $H_{(\phi)}$, and $H_{(H)}$ and consider the map defined from $W^{1,1}([-r, 0]; L^1(A_1, A_2))$ into $L^1(A_1, A_2)$ by*

$$\Gamma(\varphi)(.) = \varphi(0, .) - 2H(\mathscr{L}(\varphi))\mathscr{A}(\varphi)(.),$$
$$\text{for all } \varphi \in W^{1,1}\big([-r, 0]; L^1(A_1, A_2)\big).$$

*Then,*

(i) *$\Gamma$ is continuous from $W^{1,1}([-r, 0]; L^1(A_1, A_2))$ into $L^1(A_1, A_2)$.*

(ii) *$\mathscr{B} = \{\varphi \in W^{1,1}([-r, 0]; L^1(A_1, A_2))$: $\Gamma(\varphi)(.) = 0\}$ is a closed and nonempty set of $W^{1,1}([-r, 0]; L^1(A_1, A_2))$.*

*Proof.* (i) Let $\varphi_1, \varphi_2 \in W^{1,1}([-r, 0]; L^1(A_1, A_2))$ such that $\|\varphi_1 - \varphi_2\|_{1,1} \to 0$. We have

$$\begin{aligned}
\Gamma(\varphi_1)(x) - \Gamma(\varphi_2)(x) \\
&= \varphi_1(0, x) - \varphi_2(0, x) \\
&\quad - 2\left[H(\mathscr{L}(\varphi_1))\mathscr{A}(\varphi_1)(x) - H(\mathscr{L}(\varphi_2))\mathscr{A}(\varphi_2)(x)\right] \\
&= \varphi_1(0, x) - \varphi_2(0, x) \\
&\quad - 2H(\mathscr{L}(\varphi_1))\left[\mathscr{A}(\varphi_1)(x) - \mathscr{A}(\varphi_2)(x)\right] \\
&\quad + 2\mathscr{A}(\varphi_2)(x)\left[H(\mathscr{L}(\varphi_1)) - H(\mathscr{L}(\varphi_2))\right].
\end{aligned}$$

So,

$$\int_{A_1}^{A_2} |\Gamma(\varphi_1)(x) - \Gamma(\varphi_2)(x)|\, dx$$

$$\leq \int_{A_1}^{A_2} |\varphi_1(0, x) - \varphi_2(0, x)|\, dx$$

$$+ 2\big|H(\mathscr{L}(\varphi_1))\big| \int_{A_1}^{A_2} |\mathscr{A}(\varphi_1)(x) - \mathscr{A}(\varphi_2)(x)|\, dx$$

$$+ \big|H(\mathscr{L}(\varphi_1)) - H(\mathscr{L}(\varphi_2))\big| \int_{A_1}^{A_2} |\mathscr{A}(\varphi_2)(x)|\, dx,$$

and

$$\int_{A_1}^{A_2} |\mathscr{A}(\varphi_1)(x) - \mathscr{A}(\varphi_2)(x)|\, dx$$

$$= \int_{A_1}^{A_2} \bigg| \int_{A_1}^{A_2} \int_{\tau_1}^{\tau_2} f(x, \phi(\tau, \xi))$$

$$\times \gamma(\tau, \xi) [\varphi_1(-\tau, x) - \varphi_2(-\tau, x)]\, d\tau\, d\xi \bigg|\, dx$$

$$\leq \int_{A_1}^{A_2} \int_{\tau_1}^{\tau_2} \gamma(\tau, \xi) |\varphi_1(-\tau, x) - \varphi_2(-\tau, x)|\, d\tau\, d\xi \leq k \|\varphi_1 - \varphi_2\|_{1,1}.$$

From [10], the norm

$$\|\varphi\|_0 = |\varphi(0, .)|_{L_1} + \int_{-r}^{0} |\dot{\varphi}(\theta, .)|_{L^1}\, d\theta$$

is equivalent to $\|.\|_{1,1}$. So,

$$\int_{A_1}^{A_2} |\varphi_1(0, x) - \varphi_2(0, x)|\, dx$$

$$= \|\varphi_1 - \varphi_2\|_0 - \int_{-r}^{0} |\dot{\varphi}_1(\theta, .) - \dot{\varphi}_2(\theta, .)|_{L^1}\, d\theta.$$

We have $|\mathscr{L}(\varphi_1) - \mathscr{L}(\varphi_2)| \leq \|\varphi_1 - \varphi_2\|_{1,1}$ and $H \in \mathscr{C}^1$, so, $|H(\mathscr{L}(\varphi_1)) - H(\mathscr{L}(\varphi_2))|$, $\int_{A_1}^{A_2} |\mathscr{A}(\varphi_1)(x) - \mathscr{A}(\varphi_2)(x)|\, dx$, and $\int_1^{A_2} |\varphi_1(0, x) - \varphi_2(0, x)|\, dx$ converges to 0 as $\|\varphi_1 - \varphi_2\|_{1,1} \to 0$. Hence, the result.

(ii) The function $0 \in \mathscr{B}$ and by (i) $\Gamma$ is continuous, so $\mathscr{B} = \Gamma^{-1}(\{0\})$ is a nonempty closed set of $W^{1,1}([-r,0]; L^1(A_1, A_2))$. ▮

In order to determine an element in $\mathscr{B}$, i is sufficient to start from any function defined on $[-\tau_2, -\tau_1] \times (A_1, A_2)$. In fact, suppose $\varphi \in \mathscr{B}$: Denote $\varphi_1$ the restriction of $\varphi$ to $[-\tau_2, -\tau_1] \times (A_1, A_2)$ and $\varphi_2$ the restriction to $[-\tau_1, 0] \times (A_1, A_2)$. We search $\varphi_2 \in W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$, such that

$$\varphi_2(-\tau_1, .) = \varphi_1(-\tau_1, .) \quad \text{and} \quad \varphi_2(0, x) = k\mathscr{A}(\varphi_1)(x), \tag{30}$$

where $k = 2H(\mathscr{L}(\varphi))$. So

$$\begin{aligned} k &= 2H\left\{\int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{0}\varphi(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy\right\} \\ &= 2H\left\{\int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{-\tau_1}\varphi_1(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy\right. \\ &\qquad \left. + \int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau_1}^{0}\varphi_2(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy\right\} \\ &= 2H\left\{\int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{-\tau_1}\varphi_1(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy\right. \\ &\qquad \left. + \int_{A_1}^{A_2}\int_{-\tau_1}^{0}\varphi_2(\theta, y)\,d\theta\,dy\right\}. \end{aligned}$$

Denote

$$I = \left[\int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{-\tau_1}\varphi_1(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy, +\infty\right[.$$

If $k \in 2H(I)$, then the problem is reduced to finding $\varphi_2$, verifying (30), and such that

$$\begin{aligned} &\int_{A_1}^{A_2}\int_{-\tau_1}^{0}\varphi_2(\theta, y)\,d\theta\,dy \\ &\quad = (2H)^{-1}(k) - \int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{-\tau_1}\varphi_1(\theta, y)\,d\theta\right]\gamma(\tau, y)\,d\tau\,dy. \end{aligned} \tag{31}$$

The set of admissible $\varphi_2$ is a nonempty closed subset of $W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$. In fact, for all $\psi \in W^{1,1}([-\tau_2, -\tau_1]; L^1(A_1, A_2))$ and for $k$ in $2H(I)$, we consider the map defined from $W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$

into $\mathbb{R}$ defined by

$$F_\psi(\tilde{\psi}) = \int_{A_1}^{A_2}\int_{-\tau_1}^{0} \tilde{\psi}(\theta, y)\, d\theta\, dy - (2H)^{-1}(k)$$
$$+ \int_{A_1}^{A_2}\int_{\tau_1}^{\tau_2}\left[\int_{-\tau}^{-\tau_1} \psi(\theta, y)\, d\theta\right]\gamma(\tau, y)\, d\tau\, dy.$$

It is easily verified that $F_\psi$ is continuous on $W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$. So, $F_\psi^{-1}(\{0\})$, is a closed, nonempty subset of $W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$, contained in a ball of $L^1((-\tau_1, 0) \times (A_1, A_2))$. More precisely, it is the intersection of the ball with the positive cone of $L^1((-\tau_1, 0) \times (A_1, A_2))$. Moreover, one has to have $\varphi_2 \in W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$, with $\varphi_2(-\tau_1, .) = \varphi_1(-\tau_1, .)$ and $\varphi_2(0, .) = k\mathscr{A}(\varphi_1)(.)$. These two conditions determine a closed subset of $W^{1,1}([-\tau_1, 0]; L^1(A_1, A_2))$ which is dense in the intersection of the ball with the positive cone.

PROPOSITION 9. *Suppose $H_{(f)}$, $H_{(\gamma)}$, $H_{(\phi)}$, and $H_{(H)}$. Then, for all $n_o \geq 0$, in $\mathscr{B}$, the solution $n$ of* (1), *with $n_0$ as initial value, is positive.*

*Proof.* From the fact that the parameters $f$, $\gamma$, $\phi$, and $H$ are positive, the proof follows the same steps as in [2]. ∎

## 5. CONCLUDING REMARKS

In [2] and [4], Eq. (1), or a delay equation modelled on (1), is dealt with using a direct method. In the direct method, the equation is treated as an integral equation: One of the shortcomings of this approach is the lack of a suitable linearization that could be used in looking at the stability of solutions. The technique employed here allows the derivation of such a linearization. Developments along these lines are deferred to a further study.

## REFERENCES

1. O. Arino and M. Kimmel, Asymptotic behavior of nonlinear semigroup describing a model of selective cell growth regulation, *J. Math. Biol.* **29** (1991), 289–314.
2. O. Arino and M. Kimmel, Asymptotic behavior of nonlinear functional integral equation of cell kinetics unequal division, *J. Math. Biol.* **27** (1989), 341–354.
3. O. Arino, M. Kimmel, and M. Zerner, Analysis of a cell population model with unequal division and random transition, *Lecture Notes in Pure and Appl. Math.* **131** (1991), 3–12.
4. O. Arino and A. Mortabit, Slow oscillations in a model of cell population dynamics, *Lecture Notes in Pure and Appl. Math.* **131** (1991).

5. H. Brezis, ''Opérateurs maximaux monotones et semi-groupes de contractions sur les espaces de Hilbert,'' *in* Mathematics Studies, North Holland, Amsterdam, 1973.
6. M. G. Crandall and T. Liggett, Generation of semigroups of nonlinear transformations on General Banach Spaces, *Amer. J. Math.* **93** (1971), 265–298.
7. J. Dyson and R. Villella-Bressan, A semigroup approach to nonlinear autonomous neutral functional differential equations, *J. Differential Equations* **59**, No. 2 (1985), 206–228.
8. J. Hale, ''Functional Differential Equations,'' *in* Applied Mathematics Series, Vol. 3, Springer-Verlag, New York, 1971.
9. M. Kimmel, Z. Darzynkiewicz, O. Arino, and F. Traganos, Analysis of a cell cycle model based on unequal division of metabolic constituents to daughter cells during cytokinesis, *J. Theoret. Biol.* **110** (1984), 637–665.
10. O. Sidki, Thèse, Pau, No. 221 (1994).