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STABILITY OF A GENERAL LINEAR DELAY-DIFFERENTIAL EQUATION WITH IMPULSES.

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Abstract. In this paper we establish some sufficient conditions and also a necessary condition for asymptotic stability in a general linear delay-differential equation with impulses.
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1 Introduction

The study of certain ordinary differential equations with impulses was initiated in the $1960's$ by a seminal paper by Milman and Myshkis (see [29], [30]). After a period of active research, mostly in Eastern Europe from 1960-1970, culminating with the monograph by Halanay and Wexler [21], the subject received little attention during the seventies. An important monograph was presented by Pandit and Deo [31] in 1982, and two books by Bainov and Simeonov [9], [10], who present the state of the art in the theory of such systems. However, comparatively, not much has been done in the study of functional differential equations with impulses. In recent years, many examples of differential equations with impulses have arisen in several areas of applications and contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, seasonal changes of the water level of artificial reservoirs, as well as under the effect of floodings, can be modeled as impulses. See for example [32], [33], [35], and more specifically [18]. We study the asymptotic stability of the zero solution of our original equation, and show that under some simple conditions all solutions of the equation with impulsive effect will be asymptotically periodic. The results are transferable to the case of stationary linear compartment systems with pipes, see [19], [20].

Most of the efforts seem to have been devoted to understanding the initial value problem associated with such a system (see, e.g. Anokhin [3], Anokhin and Braverman [4]). Nonetheless, in Gopalsamy and Zhang [17], preliminary stability and oscillation results are presented for the case of a single delay under the strong condition that the delay is smaller than the length of the impulse time intervals (see also [24], [16], [1], [25], [26], [28], [34], [2], [12], [13], [5]). For some other results with Lyapunov direct method, see [8], [11].

The purpose of this paper is to investigate, as closely as possible, the question of stability for delay differential equation in the vectorial case and general delay case with impulse effects. In the vectorial case for a general distribution delay, our result extends a previous one by O. Arino and I. Gyori [6] who considered the scalar case with discrete delays. The main difficulty in this extension has been to go from discrete to distributed delays (see Remark 2.1 [6]).

We concentrate on the following problem: if the trivial solution of a delay equation is asymptotically stable without impulse effects, under what conditions can such impulsive effects maintain asymptotic stability. While some of our results can be extended to the general linear delay case and also to the case when the delay equation is nonautonomous, our main interest is in the methods and focusing the reader's attention on the role of impulses.

Section 2 gives some preliminary background, Section 3 gives a general context for this topic and is important for the investigation of both stability questions. The initial value problem is stated there. Two fundamental formulae (10)-(11) are derived for the solution of a delay differential equation with impulse effects, and its jumps via the solutions of a delay equation without impulses. Analogous formulae can be found in [3].

Section 4 deals with stability. Namely, sufficient conditions and also a necessary condition are given for asymptotic stability of system (10)-(11). Our results are given in the several delay case and we drop the strong assumption in [17] that the delay is smaller than the length of the inter-impulse time intervals. In Section 5 we provide some examples in the vectorial and scalar case and illustrate the significance and difficulties of these results.

2 Preliminaries

Let us first fix some notation:

$$\mathcal{R} = \{\varphi : [-r, 0] \to \mathbb{R}^n : \varphi \text{ has left and right limits at every points}\},$$

$\mathcal{R}$ is a regulated space, see ,

$$C = \{\varphi : [-r, 0] \to \mathbb{R}^n : \varphi(\theta) \text{ is a continuous function on } [-r, 0]\}.$$

L is a continuous linear mapping from $C([-r,0]\,;\mathbb{R}^n)$ into $\mathbb{R}^n$. By Riesz representation

$$Lx_t = \int_{-r}^{0} d\eta(\theta)x(t+\theta).$$

We will define $\widetilde{L}$ as an extension of L to $\mathcal{R}$

$$\widetilde{L}\varphi = \lim_{m\to\infty} L\varphi_m, \text{ for any } \varphi \in \mathcal{R}. \tag{1}$$

$\widetilde{L}$ is a linear operator on X defined as follows:
for any $\varphi \in \mathcal{R}$, take a sequence $\varphi_m \in C$ such that, $\lim_{m\to\infty} \varphi_m = \varphi$, with $|\varphi_m| < M$ for some $M \geq 0$.
So,

$$L\varphi_m = \int_{-r}^{0} d\eta(\theta)\varphi_m(\theta) \text{ converges}$$

and the limit is independent of the sequence.
In this paper, we consider the linear delay-differential equations with nonlinear impulses:

$$\begin{aligned} \frac{dx(t)}{dt} &= \widetilde{L}x_t, \quad x(t) \in \mathbb{R}^n, \qquad t \geq \sigma &(2)\\ x(t_j^+) - x(t_j) &= I_j(x(t_j^-)),\ x(t_j) = x(t_j^-). &(3)\end{aligned}$$

Eq. (2) is a linear delay-differential equation, (3) takes into account nonlinear impulses. We assume that $I_j \in C(\mathbb{R}^n, \mathbb{R}^n)$ (for $j = 1, 2, ...$), $(t_i)_{i\in\mathbb{Z}}$ is an increasing family of real numbers, and there exist $\delta > 0$ and $T < \infty$, such that for any $i \in \mathbb{Z}$,

$$0 < \delta \leq t_{i+1} - t_i \leq T < \infty.$$

We call (2) the impulse equation where, as usual, $x(t_j^-)$ ($x(t_j^+)$) denotes the limit from the left (from the right) of $x(t)$, as t tends to t_j, and we set the initial value problem as follows way: Let $\sigma \in \mathbb{R}$, throughout the paper, where σ denotes the initial time. Let ϕ be an element of $\mathcal{R}$. We want to find a function x defined on $[\sigma - \tau, \infty)$ such that x satisfies (2) and (3) and the initial condition:

$$x(t) = \phi(t-\sigma),\ \sigma - \tau \leq t \leq \sigma. \tag{4}$$

We consider the equation in $\mathcal{R}$, without impulses as following :

$$\begin{cases} \dfrac{du}{dt} &= \widetilde{L}u_t \\ u_0 &= \phi \in \mathcal{R} \\ u(0^+) &= x \in \mathbb{R}^n. \end{cases} \tag{5}$$

Problem (5) has, for all $(\phi, x) \in \mathcal{R} \times \mathbb{R}^n$, one and only one solution which is a regulated function on $[-r, +\infty[$.

We call T_L the semigroup associated with the equation

$$\frac{du}{dt} = Lu_t \text{ on } \mathcal{R} \times \mathbb{R}^n.$$

Thus, one has, for all $(\phi, x) \in \mathcal{R} \times \mathbb{R}^n$,

$$u_t = T_L(t)(\phi, x). \tag{6}$$

Proposition 1. *Let $(\phi, x) \in \mathcal{R} \times \mathbb{R}^n$. Then, the semigroup $T_L(t)$ associated with (5), defined by formula (6), satisfies the following relationship :*

$$T_L(t)(\phi, x) = \phi_t^0 + H_t^0 \otimes (x - \phi(0)) + (\int_0^{\max(0,\bullet)} L(T_L(s)(\phi, x))ds)_t \tag{7}$$

where

$$\phi^0(\theta) = \begin{cases} \phi(\theta) & \theta \leq 0 \\ \phi(0) & \theta > 0 \end{cases} \tag{8}$$

and H^0 is the Heaviside function

$$H^0(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases} \tag{9}$$

Proof: From equation (5), we can write the solution u in the form

$$u(t) = x + \int_0^t L(u_\tau)d\tau.$$

From (6), (8) and (9), we have

$$u_t = T_L(t)(\phi, x) = \phi_t^0 + H_t^0 \otimes (x - \phi(0)) + (\int_0^{\max(0,\bullet)} L(T_L(\tau)(\phi, x))d\tau)_t. \square$$

Definition 2. *We say that a function $x : [\sigma - r, \infty[\to \mathbb{R}^n$ is a solution of problem (2)-(4) if x is piecewise continuous on $[\sigma - \tau, \infty[$, x is differentiable on the complement of a countable subset of $[\sigma, \infty[$ and verifies (2) whenever $\frac{dx(t)}{dt}$ and the right hand side of (2) are defined, on $[\sigma, \infty[$. Finally, x has to verify the impulse equation (3) at each point $t_k, t_k \geq \sigma$, $k \geq 1$, and the initial condition (4).*

It is not difficult to show that the problem, as posed, has for each $\phi \in \mathcal{R}$ a solution. This can be readily checked by integration of (2) from σ to the first t_k on the right, then from this point to the next one and so on. The solution of (2)-(4) is denoted by $x(\sigma, \phi)$.

Throughout this paper, the solution of (2) with initial condition

$$y(s) \quad = \quad \phi(s), \quad -\tau \leq s \leq 0,$$

is denoted by $y(\phi)$. That is to say, y is a solution of the delay equation without impulses.
The fundamental solution of (2) is denoted by v, where v is defined on $[-\tau, +\infty[$, and verifies equation (2) on $]0, +\infty[$ with

$$v(0) = Id, \text{ and } v(s) = 0, \ -\tau \leq s < 0.$$

Here, denotes Id the identity matrix.
Since (2) is autonomous it is by definition that for any $(\sigma, \phi) \in \mathbb{R} \times \mathcal{R}$ the function $y(\sigma, \phi)(t) = y(\phi)(t - \sigma)$, $t \geq \sigma - r$, is the solution of problem (2) and (4). For $\sigma \in \mathbb{R}$, $k(\sigma)$ denotes the first index i such that $t_i \geq \sigma$.

3 The variation of constants formula

The next result gives a key representation formula for the solutions of (2)-(4) in terms of the solutions of (2) (without impulses).

Theorem 3. *Let $(\sigma, \phi) \in \mathbb{R} \times \mathcal{R}$. Then the solution $x(\sigma, \phi)$ of (2)-(4) can be written as*

$$x(\sigma, \phi)(t) = y(\phi)(t-\sigma) + \sum_{\sigma \leq t_j < t} v(t-t_j) u_j(\sigma, \phi), t > \sigma, \ t \notin \{t_k\}_{k \geq k(\sigma)}, \quad (10)$$

and the sequence

$$u_k(\sigma, \phi) \quad = \quad x(\sigma, \phi)(t_k^+) - x(\sigma, \phi)(t_k^-), \ k \geq k(\sigma)$$

is determined by the following nonautonomous recurrence equation

$$u_k(\sigma, \phi) = I_k(y(\phi)(t_k - \sigma) + \sum_{\sigma \leq t_j < t_k} v(t_k - t_j) u_j(\sigma, \phi)), k \geq k(\sigma), \quad (11)$$

starting from

$$u_{k(\sigma)}(\sigma, \phi) = I_{k(\sigma)}(y(\phi)(t_{k(\sigma)} - \sigma)). \quad (12)$$

Proof. Let $x = x(\sigma, \phi)$ be the solution of (2)-(4). We will show that x satisfies (10) by rewriting equation (2) in terms of a new independent variable. Namely, x can be expressed in a unique way as a sum of a continuous function $z : [\sigma, \infty) \to \mathbb{R}^n$ and step functions whose jumps are at the points t_k and are equal to the jumps of x at these points.

In terms of z and the $u_k = u_k(\sigma, \phi)'s$

$$x(t) \quad = \quad z(t) + \sum_{\sigma \leq t_j < t} H^0(t - t_j) u_j, \ t \geq \sigma, \quad (13)$$

$$x(t) \quad = \quad z(t), \ t \leq \sigma, \quad (14)$$

where H^0, denotes the Heaviside function.
The function $z(t)$ introduced above is continuous on $[\sigma-\tau,+\infty[$ and differentiable where x is differentiable. Moreover, z satisfies,

$$\begin{aligned} \frac{dz(t)}{dt} &= L(z_t)+\sum_{\sigma\le t_j<t}\widetilde{L}(H^0_{t-t_j}u_j),\ t>\sigma \qquad (15)\\ z(t) &= \phi\left(t-\sigma\right),\ \sigma-\tau\le t\le\sigma. \qquad (16)\end{aligned}$$

Equation (15)-(16) is an inhomogeneous equation of the form

$$\begin{aligned} \frac{dz(t)}{dt} &= L(z_t)+f(t), \quad t>\sigma\\ z(t) &= \phi\left(t-\sigma\right), \quad \sigma-\tau\le t\le\sigma\end{aligned}$$

where the inhomogeneous term $f(t)=\sum_{\sigma\le t_j<t}\widetilde{L}(H^0_{t-t_j}u_j)$ is locally integrable on $[\sigma,\infty)$. Recall (see [22]) that the solution of the inhomogeneous equation can be expressed in terms of the solution of the homogenous equation and the inhomogeneous term:

$$z(t)=y(\phi)(t-\sigma)+\int_\sigma^t v(t-s)\sum_{\sigma\le t_j<s}\widetilde{L}(T_{t_j}H^0_s u_j)ds,\ t\ge\sigma. \qquad (17)$$

We remark that in our case the inhomogeneous term is just locally integrable. But, it is known that the inhomogeneous equation makes sense and yields

$$z(t)=y(\phi)(t-\sigma)+\sum_{\sigma\le t_j<t}\int_\sigma^t v(t-s)\widetilde{L}(T_{t_j}H^0_s u_j)ds,\ t\ge\sigma. \qquad (18)$$

Substituting the right hand side of the above relation for z in formula (13), we arrive at a first expression for x :

$$x(t)=y(\phi)(t-\sigma)+\sum_{\sigma\le t_j<t}[u_j+\int_\sigma^t v(t-s)\widetilde{L}(T_{t_j}H^0_s u_j)ds],\ t\ge\sigma. \qquad (19)$$

We will now get a relation involving the $u_k's$. For this, note that taking the limit from the left of formula (13) as t tends to $t_k>\sigma$, we have

$$x(t_k^-)=z(t_k)+\sum_{\sigma\le t_j<t_k}u_j. \qquad (20)$$

Taking formula (18) at $t=t_k$, we obtain

$$z(t_k)=y(\phi)(t_k-\sigma)+\sum_{\sigma\le t_j<t_k}\int_\sigma^{t_k} v(t_k-s)\widetilde{L}(H^0_s u_j)ds. \qquad (21)$$

Substituting the right hand side of the latter expression for $z(t_k)$ in the expression of $x(t_k^-)$, and using equation (3), we arrive at

$$u_k = I_k(y(\phi)(t_k - \sigma) + \sum_{\sigma \le t_j < t_k} [u_j + \int_{\sigma}^{t_k} v(t_k - s)\widetilde{L}(H^0_{s-t_j} u_j)ds]), \text{ for } t_k > \sigma. \tag{22}$$

In particular, if $k = k(\sigma)$, then

$$\{j : \sigma \le t_j < t_{k(\sigma)}\} = \emptyset,$$

and therefore

$$u_{k(\sigma)} = I_{k(\sigma)}(y(\phi)(t_{k(\sigma)} - \sigma)),$$

which yields formula (12).
By Riesz representation, and (1), we have

$$\widetilde{L}(H^0_{s-t_j} u_j) = \int_{-\tau}^{0} d\eta(\theta) H^0_{s-t_j}(\theta) u_j.$$

Now we simplify both formulas (19) and (22), for $\sigma \le t_j < t$.
In fact by Fubini's theorem and using the facts that $v(t) = 0$ for $t < 0$, $t_j \ge \sigma$, where $H^0(t)$ is the Heaviside function,
we have

$$\begin{aligned}
\int_{\sigma}^{t} v(t-s)\widetilde{L}(H^0_{s-t_j} u_j)ds &= \int_{\sigma}^{t} (\int_{-r}^{0} v(t-s) d\eta(\theta) H^0(s - t_j + \theta) u_j) ds \\
&= \int_{-r}^{0} (\int_{\sigma - t_j + \theta}^{0} v(t - s - t_j + \theta) H^0(s) d\eta(\theta) u_j ds) \\
&\quad + \int_{0}^{t - t_j + \theta} v(t - s - t_j + \theta) H^0(s) d\eta(\theta) u_j ds) \\
&= \int_{0}^{t - t_j} (\int_{-r}^{0} v(s + \theta) d\eta(\theta) u_j) ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \left(\int_{-r}^0 v(s+\theta)d\eta(\theta)u_j\right)ds &= \int_{-r}^0 \left(\int_0^{t+\theta} v(s)d\eta(\theta)u_j\right)ds \\
&= \left(\int_0^t v(s)ds\right)\left(\int_{-r}^0 d\eta(\theta)u_j\right) - \\
&\quad \int_0^t v(t-s)ds)\left(\int_{-r}^{-s} d\eta(\theta)u_j\right) \\
&= \left(\int_0^t v(s)ds\right)\left(\int_{-r}^0 d\eta(\theta)u_j\right) + \\
&\quad \left(\int_0^t v(s)ds\right)\left(\int_0^{-r} d\eta(\theta)u_j\right) + \\
&\quad \left(\int_0^t v(t-s)ds\right)\left(\int_{-s}^0 d\eta(\theta)u_j\right) \\
&= \left(\int_0^t v(t-s)ds\right)\left(\int_{-s}^0 d\eta(\theta)u_j\right).
\end{aligned}$$

But from the definition of the fundamental solution v one can easily see that

$$v(t) = Id + \int_0^t v(t-s)L(\varphi_s^0)ds,$$

where

$$\varphi^0(t) = \begin{cases} Id & t \geq 0 \\ 0 & t < 0 \end{cases}$$

we then, have

$$\begin{aligned}
v(t) &= Id + \int_0^t v(t-s)\left(\int_{-r}^0 d\eta(\theta)\varphi_s^0(\theta)\right)ds \\
&= Id + \int_0^t v(t-s)\left(\int_{-s}^0 d\eta(\theta)\right)ds,
\end{aligned}$$

and

$$v(t-t_j) = Id + \int_{-r}^0 \int_0^{t-t_j} v(s+\theta)d\eta(\theta)ds, \quad t > t_j > \sigma.$$

Substituting the above expression for the coefficients of the $u_j's$ in (19) and (22), we arrive at the formula

$$x(\sigma,\phi)(t) = y(\phi)(t-\sigma) + \sum_{\sigma \leq t_j < t} v(t-t_j)u_j(\sigma,\phi), \ t \geq \sigma,$$

and

$$u_k(\sigma,\phi) = I_k(y(\phi)(t_k-\sigma) + \sum_{\sigma \leq t_j < t_k} v(t_k-t_j)u_j(\sigma,\phi)), \text{ for } t_k > \sigma.$$

The proof of Theorem 3 is complete.□

The stability issue for the linear delay-differential equations with nonlinear impulses (2)-(4) is examined in the next section from the view point of equation (2) perturbed by the impulse equation (3).

4 Stability

Assuming that I_j is linear, that Eq. (2) is asymptotically stable and thus all the solutions of (2) (without impulses) tend to zero as t tends to $+\infty$, we set the following question: Under which additional condition is the linear delay-differential equation with nonlinear impulses (2)-(4) asymptotically stable? Let $B(t,\sigma)$ be defined by

$$B(t,\sigma) = \sum_{\sigma \le t_j < t} \|v(t-t_j)\| \, \|I_j\|_\infty \,, \; t > \sigma, \tag{23}$$

where v is the fundamental solution of (2), and

$$\|I_j\|_\infty = \sup_{x \neq 0} \frac{|I_j(x)|}{x}, \;\; \|v(t)\| = \sup_{\|x\| \le 1} |v(t)x| \,, \text{ with } x \in \mathbb{R}^n.$$

Before considering our basic stability result, we need to introduce the following lemma.

Lemma 4. *Assume that (2) (without impulses) is asymptotically stable. Then*
(α) If there exists $\sigma_1 \in \mathbb{R}$ such that

$$\sup_{t \ge \sigma_1} B(t,\sigma_1) < \infty,$$

then

$$A(\sigma) := \sup_{t \ge \sigma} B(t,\sigma) < \infty, \qquad \sigma \in \mathbb{R}.$$

(β) For any $\sigma \in \mathbb{R}$, one has

$$\overline{A}_0 := \lim_{k \to +\infty} \sup B(t_k, 0) = \lim_{k \to +\infty} \sup B(t_k,\sigma),$$

and

$$\underline{A}_0 := \lim_{k \to +\infty} \inf B(t_k, 0) = \lim_{k \to +\infty} \inf B(t_k,\sigma).$$

Proof. By the definition of $B(t,\sigma)$, it is clear that

$$B(t,\sigma_2) = \sum_{\sigma_2 \le t_j < \sigma_3} \|v(t-t_j)\| \, \|I_j\|_\infty + B(t,\sigma_3)$$

for any $-\infty < \sigma_2 < \sigma_3 < t < \infty$. But equation (2) is asymptotically stable and hence v is bounded on $[0, \infty)$ and it tends to zero as t tends to $+\infty$. Thus

$$\sup_{t \geq \sigma_2} \sum_{\sigma_2 \leq t_j < \sigma_3} \|v(t - t_j)\| \, \|I_j\|_\infty < \infty,$$

and

$$\sum_{\sigma_2 \leq t_j < \sigma_3} \|v(t - t_j)\| \, \|I_j\|_\infty \to 0 \text{ as } t \to +\infty$$

which implies the statements of the lemma.□
We now state and prove our basis stability result.

Theorem 5. *Assume that (2) (without impulses) is asymptotically stable, that is, all the solutions tend to zero as t tends to $+\infty$, moreover*

$$\lim_{t \to +\infty} \sup B(t, 0) < \infty \text{ and } \lim_{k \to +\infty} \sup B(t_k, 0) < 1, \tag{24}$$

then for any $(\sigma, \phi) \in \mathbb{R} \times \mathcal{R}$ the solution $x(\sigma, \phi)$ of problem (2)-(4) tends to zero $t \to +\infty$.

Proof. Let $(\sigma, \phi) \in \mathbb{R} \times \mathcal{R}$ be fixed and consider the solution $x = x(\sigma, \phi)$ of the initial value problem (2)-(4). Then we have that x satisfies (10) and (11), where $y(\phi)(t - \sigma)$ is a solution of the delay equation without impulses, and $v(t)$ is the fundamental solution of (2). But (2) is autonomous and asymptotically stable and hence there exists a constant $c_1 > 0$ independent on (σ, ϕ) such that

$$|y(\phi)(t - \sigma)| \leq c_1 \|\phi\|, \quad t \geq \sigma, \tag{25}$$

where $\|\phi\| = \sup_{-r \leq s \leq 0} |\phi(s)|$. Moreover

$$\lim_{t \to +\infty} |y(\phi)(t - \sigma)| = 0 \quad \text{and} \quad c_2 = \sup_{t \geq 0} \|v(t)\| < \infty. \tag{26}$$

On the other hand by (25) and lemma 4, we may fix a constant $c \geq 0$ such that

$$A_0 := \sup_{u \in \mathbb{R}} \left\{ \sup_{k \geq k(\max\{u,c\})} B(t_k, u) \right\} < 1,$$

where $u \in \mathbb{R}$.
Define:

$$M = \max\{\tfrac{1}{1-A_0}(c_1 \|\phi\| + c_2 \sum_{k(\sigma) \leq j \leq k(\max\{\sigma,c\})} |u_j|), \max_{k(\sigma) \leq k \leq k(\max\{\sigma,c\})} \left|I_k^{-1}(u_k)\right|\}. \tag{27}$$

Then

$$\left|I_{k(\sigma)}^{-1}(u_{k(\sigma)})\right| \leq M \tag{28}$$

and by induction we show that

$$\left|I_j^{-1}(u_j)\right| \le M, \qquad j \ge k(\sigma). \tag{29}$$

Indeed, if (29) is satisfied for $k(\sigma) \le j \le i$, then (11) yields

$$\begin{aligned}
\left|I_{i+1}^{-1}(u_{i+1}(\sigma,\phi))\right| &= \left| y(\phi)(t_{i+1}-\sigma) + \sum_{\sigma \le t_j < t_{i+1}} v(t_{i+1}-t_j)u_j(\sigma,\phi) \right| \\
&\le c_1 \|\phi\| + \sum_{k(\sigma)\le j \le k(\max\{\sigma,c\})} \|v(t_{i+1}-t_j)\| \, |u_j(\sigma,\phi)| \\
&\quad + \sum_{k(\max\{\sigma,c\})<j\le i} \|v(t_{i+1}-t_j)\| \, \|I_j\|_\infty \left|I_j^{-1}(u_j(\sigma,\phi))\right| \\
&\le c_1\|\phi\| + c_2 \sum_{k(\sigma)\le j\le k(\max\{\sigma,c\})} |u_j(\sigma,\phi)| + A_0 M \\
&\le M,
\end{aligned}$$

and therefore (29) holds for all $j \ge k(\sigma)$. Thus we have

$$K = \lim_{k\to+\infty} \sup \left|I_k^{-1}(u_k)\right| < \infty.$$

Let $\varepsilon > 0$ be fixed. Then there exists $T_1 = T_1(\varepsilon) \ge 0$ such that

$$\left|I_j^{-1}(u_j)\right| < K + \varepsilon, \quad t_j \ge \sigma + T_1.$$

On the other hand, under condition (24), by lemma 4, we have

$$\overline{A}_0 = \lim_{k\to+\infty} \sup \sum_{\sigma+T_1\le t_j<t_k} \|v(t_k-t_j)\| \, \|I_j\|_\infty < 1.$$

Since I_k is invertible, and using (11), we see that

$$\begin{aligned}
K &= \lim_{k\to+\infty} \sup \left|I_k^{-1}(u_{k)})\right| \\
&\le \lim_{k\to+\infty} \sup |y(\phi)(t_k-\sigma)| + \lim_{k\to+\infty} \sup \sum_{\sigma\le t_j<\sigma+T_1} \|v(t_k-t_j)\| \, |u_j| \\
&\quad + \lim_{k\to+\infty} \sup \sum_{T_1+\sigma\le t_j<t_k} \|v(t_k-t_j)\| \, \|I_j\|_\infty \left|I_j^{-1}(u_{j)})\right| \\
&\le (K+\varepsilon) \lim_{k\to+\infty} \sup \sum_{T_1+\sigma\le t_j<t_k} \|v(t_k-t_j)\| \, \|I_j\|_\infty = \overline{A}_0 K + \overline{A}_0 \varepsilon.
\end{aligned}$$

Therefore,

$$0 \le K \le \frac{\overline{A}_0}{1-\overline{A}_0}\varepsilon.$$

This yields that

$$\lim_{k\to+\infty} I_k^{-1}(u_k)=0.$$

Now, we show that $\lim\limits_{t\to+\infty} x(t)=0$. There exists $T_2=T_2(\varepsilon)\geq 0$, such that

$$\left|I_j^{-1}(u_j)\right|<\varepsilon,\quad t_j\geq \sigma+T_2.$$

In that case, using (10), we find

$$\begin{aligned}\lim_{t\to+\infty}\sup|x(t)| &\leq \lim_{t\to+\infty}\sup|y(\phi)(t-\sigma)|+\lim_{t\to+\infty}\sup\sum_{\sigma\leq t_j<\sigma+T_2}\|v(t_k-t_j)\|\,|u_j| \\ &\quad+\lim_{t\to+\infty}\sup\sum_{T_2+\sigma\leq t_j<t}\|v(t-t_j)\|\,\|I_j\|_\infty\left|I_j^{-1}(u_j)\right| \\ &\leq \varepsilon\lim_{t\to+\infty}\sup\sum_{T_2+\sigma\leq t_j<t}\|v(t-t_j)\|\,\|I_j\|_\infty\,.\end{aligned}$$

We use condition (24) and lemma 4, to find

$$\lim_{t\to+\infty}\sup\sum_{T_2+\sigma\leq t_j<t}\|v(t-t_j)\|\,\|I_j\|_\infty=A<\infty.$$

Since $\varepsilon>0$ was arbitrarily fixed,

$$\lim_{t\to+\infty}\sup|x(t)|\leq\varepsilon A,$$

and this yields $\lim\limits_{t\to+\infty} x(t)=0$. The proof is complete.□

The solution of the initial value problem (2)-(4) is uniformly stable on $[0,\infty)$. That is, for any $\varepsilon>0$ there exits $\delta=\delta(\varepsilon)>0$ such that for any $(\sigma,\phi)\in\mathbb{R}_+\times\mathcal{R}$ the solution $x(\sigma,\phi)$ of the system (2)-(4) satisfies

$$|x(\sigma,\phi)(t)|<\varepsilon,\ t\geq\sigma,\ \text{whenever}\ \|\phi\|=\sup_{-r\leq s\leq 0}|\phi(s)|<\delta.$$

We have this proposition.

Proposition 6. *Assume that I_j is invertible, equation (2) is asymptotically stable and (24) is satisfied. Then there exists a constant K such that for any $(\sigma,\phi)\in\mathbb{R}_+\times\mathcal{R}$ the solution $x(\sigma,\phi)$ of system (2)-(4) satisfies*

$$|x(\sigma,\phi)(t)|\leq K\|\phi\|,\ t\geq\sigma, \tag{30}$$

and hence the zero solution of (2)-(4) is uniformly stable on $[0,\infty)$.

Proof. Let $(\sigma,\phi)\in\mathbb{R}_+\times\mathcal{R}$ be fixed and consider the solution $x=x(\sigma,\phi)$ of (2)-(4). Then using (12) and (25) we obtain

$$\left|I_{k(\sigma)}^{-1}(u_{k(\sigma)}(\sigma,\phi))\right|\leq c_1\|\phi\|,\qquad t\geq\sigma.$$

That is, $u_{k(\sigma)} = u_{k(\sigma)}(\sigma, \phi)$ satisfies

$$\left| I^{-1}_{k(\sigma)}(u_{k(\sigma)}) \right| \le c_1 \|\phi\|,$$

and by induction we show that

$$\left| I^{-1}_j(u_j) \right| \le c_3 \|\phi\|, \quad j \ge k(\sigma).$$

Combining (11), (25) and (26), it can be easily seen that for $k(\sigma) \le j < i$

$$\left| I^{-1}_{i+1}(u_{i+1}(\sigma, \phi)) \right| \le c_1 \|\phi\| + c_2 \sum_{k(\sigma) \le j < i+1} |u_j|.$$

Thus, (27) and (29) yield

$$\left| I^{-1}_j(u_j) \right| \le c_3 \|\phi\|, \qquad j \ge k(\sigma),$$

where the constant c_3 does not depend on σ. But, in that case, from (10) it follows

$$|x(\sigma, \phi)(t)| \le c_1 \|\phi\| + c_2 c_3 \|\phi\| = K \|\phi\|, \qquad \sigma \ge 0,$$

where $K = c_1 + c_2 c_3$ is independent of σ. The proof of the proposition is complete.□

5 Example and result

In our consideration, equation (2) is asymptotically stable. This implies that the exponential growth rate of the fundamental solution v of (2) and (4) is negative [22]. More precisely there exist constants $M \ge 1$ and $\omega > 0$ such that

$$\|v(t)\| \le M \exp(-\omega(t - \sigma)), \; t \ge \sigma \ge 0. \tag{31}$$

Example : We consider the delay differential equation with impulses

$$\frac{dx(t)}{dt} = Nx(t-1), \; t > 0, \; x \in \mathbb{R}^2, \tag{32}$$

$$x_0(t) = \phi(t), \; -1 \le t \le 0, \tag{33}$$

$$x(t_j^+) - x(t_j) = \beta_j(x(t_j)), \; x(t_j) = x(t_j^-), \tag{34}$$

where $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in \mathbb{R}$, $\beta_j \in \mathbb{R}^2$ the characteristic equation is

$$P(\lambda) = \det(\lambda I - Ne^{-\lambda}) = \det(\lambda e^{\lambda} I - N) = 0$$

and we have

$$y^2 - (a+d)y + ad - cb = 0, \text{ where } y = \lambda e^{\lambda}.$$

Then,

$$y = \lambda e^{\lambda} = \tfrac{(a+d)\pm\sqrt{(a+d)^2-4(ad-cb)}}{2}, \text{ with } (a+d)^2 - 4(ad-cb) \geq 0.$$

$\lambda = -1$ is a root **iff** the following relation is true

$$\frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2} = -e^{-1} \tag{35}$$

and we have

$$\begin{aligned} \Delta(\lambda) &= \lambda I - Ne^{-\lambda} = \begin{bmatrix} \lambda - ae^{-\lambda} & -be^{-\lambda} \\ -ce^{-\lambda} & \lambda - de^{-\lambda} \end{bmatrix} \\ \Delta^{-1}(\lambda) &= \frac{adj\Delta(\lambda)}{\det \Delta(\lambda)}. \end{aligned}$$

We calculate $\det \Delta(\lambda)$:

$$\det \Delta(\lambda) = (\lambda + e^{-(\lambda+1)})F(\lambda)$$

where

$$F(\lambda) = (\lambda - qe^{-\lambda}), \ q = \tfrac{(a+d)-\sqrt{(a+d)^2-4(ad-cb)}}{2}, \text{ and } F(-1) \neq 0.$$

From the Taylor series we obtain

$$\Delta^{-1}(\lambda) = \frac{2(1 + \frac{1}{3}(\lambda+1) + o((\lambda+1)^2))}{(\lambda+1)^2 F(\lambda)} \begin{bmatrix} \lambda - de^{-\lambda} & be^{-\lambda} \\ ce^{-\lambda} & \lambda - ae^{-\lambda} \end{bmatrix}$$

and the fundamental solution is given by this formula, (see [23], [15])

$$v(t) = Res(e^{\lambda t}\Delta^{-1}(\lambda); \lambda = -1) = \lim_{\lambda \to -1} \frac{d}{d\lambda}((\lambda+1)^2 e^{\lambda t}\Delta^{-1}(\lambda)).$$

After some calculation we have

$$v(t) = M_t w e^{-t}, \text{ where } M_t = 2(t + \frac{1}{3}), \ w = \begin{bmatrix} \frac{1+de}{2+(a+d)e} & \frac{-be}{2+(a+d)e} \\ \frac{ce}{2+(a+d)e} & \frac{1+ae}{2+(a+d)e} \end{bmatrix}.$$

Theorem 7. *Assume that (35) is satisfied and*

$$\lim_{j \to +\infty} \sup \|\beta_j\|_{\infty} < \frac{1}{M_T \|w\|}(\exp(\delta) - 1), \ M_T \textit{ is a constant.} \tag{36}$$

Then for any $(\sigma, \phi) \in \mathbb{R} \times \mathcal{R}$ the solution $x(\sigma, \phi)$ of the initial value problem (32)-(34) tends to zero as $t \to +\infty$.

Proof. From (36) it follows that there exist a constant $\|I\|_\infty$ and index j_0 such that

$$\|\beta_j\|_\infty \leq \|\beta\|_\infty \ , \ j \geq j_0$$

where

$$\|\beta_j\|_\infty = \sup_{x \neq 0} \frac{|\beta_j(x)|}{|x|}$$

and

$$\|\beta\|_\infty < \frac{1}{M_T \|w\|}(e^\delta - 1).$$

Moreover, since $t_j \to \infty$ $(j \to \infty)$, we may fix σ_0 such that $k(\sigma_0) \geq j_0$. Thus from (23) and (31), it follows that

$$\begin{aligned} |B(t_k, \sigma_0)| &\leq M_{t_k - t_j} \|w\| \sum_{\sigma_0 \leq t_j < t} e^{-(t_k - t_j)} \|\beta_j\|_\infty \\ &\leq M_T \|w\| \sum_{j=k(\sigma_0)}^{k-1} (\exp(-\delta))^{k-j} \|\beta\|_\infty \\ &\leq M_T \|w\| \|\beta\|_\infty e^{-\delta}((\exp(-\delta))^{k-k(\sigma_0)} - 1)\frac{1}{e^\delta - 1}, \end{aligned}$$

for all $k > k(\sigma_0)$. Thus,

$$\lim_{k \to +\infty} \sup |B(t_k, \sigma_0)| \leq M_T \|w\| \|\beta\|_\infty \frac{e^{-\delta}}{1 - e^{-\delta}} = M_T \|w\| \|\beta\|_\infty \frac{1}{e^\delta - 1} < 1. \quad (37)$$

Moreover, for $t > \sigma_0$, one can find

$$|B(t, \sigma_0)| \leq M_T \|w\| \sum_{\sigma_0 \leq t_j < t} e^{-(t - t_j)} \|\beta_j\|_\infty \leq M_T \|w\| \|\beta\|_\infty \sum_{j=k(\sigma_0)}^{k(t)} e^{-(t_{k(t)} - t_j)}$$

where $k(t)$ denotes the greatest index for which $t_k < t$. Thus, (37) yields that

$$\lim_{t \to +\infty} \sup |B(t, \sigma_0)| < \infty,$$

and hence all the conditions in theorem 5 are satisfied. The proof of Theorem 7 is completed by using theorem 5.

The main problem when applying condition (24) is that we do not know precisely the values of M and ω.
Let us consider the characteristic equation associated with the **scalar case** of delay differential equation without impulses (2) and (4)

$$\lambda = \int_{-r}^{0} e^{\lambda_1 \theta} d\eta(\theta). \quad (38)$$

We deduce the following lemma.

Lemma 8. *Assume that the fundamental solution v of (2) is positive on $[0,\infty)$ and the leading (real) root λ_1 of (38) satisfies*

$$\int_{-r}^{0} e^{\lambda_1\theta}\,|\theta|\,(d\eta(\theta))^- < 1 \tag{39}$$

where $(d\eta(\theta))^- = \max\{0, -d\eta(\theta)\}$. Then

$$v(t) \le \frac{1}{1-\int_{-r}^{0} e^{\lambda_1\theta}\,|\theta|\,(d\eta(\theta))^-} e^{-\lambda_1 t},\ t \ge 0. \tag{40}$$

Proof. Let

$$y(t) = v(t)e^{-\lambda t},\ t \ge -r.$$

Then, for any $t \ge 0$, we find

$$\begin{aligned}
\dot{y}(t) &= (\int_{-r}^{0} d\eta(\theta)v(t+\theta))e^{-\lambda_1 t} - \lambda_1 y(t) \\
&= \int_{-r}^{0} d\eta(\theta)e^{\lambda_1\theta}(y(t+\theta)-y(t)).
\end{aligned}$$

But $y(t) = 0$, $-r \le t < 0$, $y(0) = 1$, and hence integrating both sides of the above equation we obtain

$$\begin{aligned}
y(t) &= 1 + \int_0^t (\int_{-r}^{0} d\eta(\theta)e^{\lambda_1\theta}(y(s+\theta)-y(s))ds \\
&= 1 + \int_{-r}^{0} d\eta(\theta)e^{\lambda_1\theta}(\int_0^{t+\theta} y(s)ds - \int_0^t y(s)ds) \\
&= 1 + \int_{-r}^{0} d\eta(\theta)e^{\lambda_1\theta}(\int_t^{t+\theta} y(s)ds),\ \text{for } t \ge 0.
\end{aligned}$$

Now we show that

$$y(t) < \frac{1}{1-\int_{-r}^{0} e^{\lambda_1\theta}\,|\theta|\,(d\eta(\theta))^-},\ \text{for } t \ge 0.$$

Moreover, $y(0) = 1$, then (40) holds for $t = 0$.
Otherwise, since $y(t)$ is continuous, there exists $T > 0$ such that

$$y(t) < \frac{1}{1-\int_{-r}^{0} e^{\lambda_1\theta}|\theta|(d\eta(\theta))^-},\ 0 \le t < T,\ \text{and } y(T) = \frac{1}{1-\int_{-r}^{0} e^{\lambda_1\theta}|\theta|(d\eta(\theta))^-}.$$

In that case

$$\begin{aligned}
y(T) &\le 1 + \int_{-r}^{0} (d\eta(\theta))^- e^{\lambda_1\theta}(\int_{T+\theta}^{T} y(s)ds) \\
&\le 1 + \frac{\int_{-r}^{0}(d\eta(\theta))^-\,|\theta|\,e^{\lambda_1\theta}}{1-\int_{-r}^{0} e^{\lambda_1\theta}\,|\theta|\,(d\eta(\theta))^-} \\
&< \frac{1}{1-\int_{-r}^{0} e^{\lambda_1\theta}\,|\theta|\,(d\eta(\theta))^-},
\end{aligned}$$

which is a contradiction. Thus

$$v(t)e^{-\lambda_1 t} < \frac{1}{1 - \int_{-r}^{0} e^{\lambda_1 \theta} |\theta| (d\eta(\theta))^{-}}, \ t \geq 0,$$

and the proof of the lemma is complete.□

The following result is an easy consequence of Lemma 8.

Theorem 9. *Assume that (2)* ***-in the scalar case-*** *is asymptotically stable, and (39) is satisfied. If*

$$\lim_{j \to +\infty} \sup \|I_j\|_{\infty} < (1 - \int_{-r}^{0} e^{\lambda_1 \theta} |\theta| (d\eta(\theta))^{-})(\exp(-\lambda_1 \delta) - 1)$$

then, the conclusion of Theorem 5 is valid.

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