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Integrated semigroup and linear ordinary differential equation with impulses.

M.Bachar
Institute for Mathematics
Heinrichstraße 36
8010 Graz, Austria
mostafa.bachar@uni-graz.at

O. Arino
GEODES-IRD 32,
av. Henri Varagnat
F-93143 Bondy cedex, France
arino@bondy.ird.fr

Abstract. In this paper, we discuss the fundamental linear theory for ordinary differential equations with impulses. We show, using the general theory of integrated semigroups, that we can associate a strongly continuous semigroup with any ordinary differential equation with impulses.

1 Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis (see [22], [23]). They formalized the situation when the state of a system changes as a result of jumps occurring at different moments of time. The times at which jumps occur may be known and form a sequence of times with or without a certain pattern, or may be determined in terms of the state itself. Examples of equations with impulses can be found in various contexts: in the periodic treatment of some diseases, impulses correspond to administration of a drug or a missing product; in environmental sciences, seasonal changes of the water level of artificial reservoirs, as well as under the effect of floodings, can be modeled as impulses. Ordinary differential equations with impulses have already been considered extensively in the literature (see the monographs [15], [31]). In the recent years, differential equations with impulses have flourished in several contexts, notably in the modeling of the effects of repeated drug treatment, see [32]. In this paper we

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consider ordinary differential equations with impulses. By an impulse, we mean a sudden change of the state of a system: at each moment of a possibly unbounded sequence of moments, the state jumps from one position to another, as a consequence of a transformation which depends only on the moment of the impulses. We remark that the problem with impulses is no more an autonomous problem., the prototype of ordinary differential equation with impulses as follows:

$$\begin{cases} \dfrac{du}{dt} = Au(t),\ t > \sigma,\ t \neq t_i,\ i \in \mathbb{Z},\ \sigma \in \mathbb{R}\ (DE) \\ u(\sigma) = \xi \in X,\ (IC) \\ u(t_i^+) = B_i u(t_i^-),\ u(t_i^-) = u(t_i),\ t_i \geq \sigma,\ i \in \mathbb{Z},\ (IMC) \end{cases} \tag{1.1}$$

X is any Banach space. The operator A is a bounded linear operator; the last equation introduces the jumps, which make necessary to working on a natural space in the context of impulses, which is the space of regulated -we say that the function f is regulated if f has left and right limits at every point (the limit here is not uniform but only a pointwise limit)-, and we denote by $\lim\limits_{<}$ resp. $\lim\limits_{>}$, the **pointwise limit** to the right, resp. to the left, see [8], [12]. Our purpose is to provide a linear theory for such equations in Banach spaces. There are two main challenges: the first one is set by the jump discontinuities which make necessary to extend the usual state space of continuous functions to a space of functions having some discontinuities; the second one is the time-dependence of the system, arising implicitly from the time jumps. The method used to overcome these two problems is two-fold:
1) Time-dependence will be eliminated by a recurse to extrapolation theory, see [10] [21] [24] [30] [26] [28] [27] [38].
2) Integration will be used to smooth down the discontinuities. This goes through the now well-known integrated semigroup theory, see [1] [2] [3] [4] [5] [6] [7].

We will now describe the main results and the main steps to be accomplished in order to derive these results. Throughout the paper, we denote $U(t, s)$ the evolution operator which maps initial values, given at time s, to the solution at any future time t, and $T(t)$ the operator defined as following

$$(T(t)f)(s) = U(s, s-t)(f(s-t)), \tag{1.2}$$

where $f \in BR(\mathbb{R}, X)$, $s \in \mathbb{R}$, $t \geq 0$, where $BR(\mathbb{R}, X)$ is the space of bounded regulated functions $\mathbb{R} \to X$ continuous to the left.
The operator $T(t)$ defined by formula (1.2) associated with delay differential equations with impulses (1.1), in subsection 5.1, can be writhed as:
⧫If $[s-t, s[\cap D = \emptyset$

$$(T(t)f)(s) = e^{tA} f(s-t), \tag{1.3}$$

⧫If $[s-t, s[\cap D = \{t_n\}$

$$(T(t)f)(s) = e^{(s-t_i)A} \circ B_i \circ e^{(t_i-(s-t))A} f(s-t), \tag{1.4}$$

⧫If $[s-t, s[\cap D = \{t_i, i = n, n+1, ..., m; m > n, (n, m) \in \mathbb{Z}^2\}$

$$(T(t)f)(s) = e^{(s-t_m)A} \circ B_m \circ e^{(t_m-t_{m-1})A} \circ ... \circ B_n \circ e^{(t_n-(s-t))A} f(s-t). \tag{1.5}$$

Our first result states that :

Theorem 1.1 $(T(t))_{t\geq 0}$ *is a pointwise regulated semigroup of bounded linear operators on* $BR(\mathbb{R}, X)$.

Then, we introduce the following family of operators

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau, \tag{1.6}$$

for $f \in BR(\mathbb{R}, X)$, $s \in \mathbb{R}$, $t \geq 0$, and we have the followings theorem:

Theorem 1.2 $(S(t))_{t>0}$ *defined by formula (1.6) is a norm continuous integrated semigroup on* $BR(\mathbb{R}, X)$.

An important feature revealed by next Theorem 1.3 is the fact that the integrated semigroup takes its values in the space of functions whose discontinuities are concentrated in the set D the set of times of jumps. This weak regularizing property is the analog of what happens in integrated semigroups.

Theorem 1.3 *Let* S *be given by (1.6), and* $f \in BR(\mathbb{R}, X)$*. Then,* $s \to (S(t)f)(s)$ *is continuous at each* $s \notin \{t_i\}$ *and all* $t > 0$ *fixed, and we have*

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s) = B_i \lim_{\substack{s \to t_i \\ <}} (S(t)f)(s).$$

Finally, Theorem 1.4 describes the infinitesimal generator associated with the semigroup $T(t)$:

Theorem 1.4 *The operator* $\mathcal{A}$ *defined by*

$$D(\mathcal{A}) = \left\{ \begin{array}{l} f \in BR(\mathbb{R}, X) : \ singf \subset \{t_i\}_{i \in \mathbb{Z}}, \ \lim\limits_{\substack{s \to t_i \\ >}} f(s) = \lim\limits_{\substack{s \to t_i \\ <}} B_i f(s), \\ and \ \dfrac{\partial f}{\partial s} \in BR(\mathbb{R}, X), \ sing\dfrac{\partial f}{\partial s} \subset \{t_i\}_{i \in \mathbb{Z}}, \end{array} \right\}.$$

$$(\mathcal{A}f)(s) = Af(s) - f^{'}(s)$$

and, we suppose that

$$\sup_{i \in \mathbb{Z}} \|B_i\| < e \tag{1.7}$$

is the generator of locally Lipschitz continuous integrated semigroup $S(t)$ *on* $BR(\mathbb{R}, X)$*, which satisfies* $S(t)(BR(\mathbb{R}, X)) \subset C(\mathbb{R} - \{t_i\}_{i \in \mathbb{Z}}, X)$ *and*

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau, \ for \ f \in BR(\mathbb{R}, X),$$

where $T(t)$ *is defined by (1.3)-(1.5).*

2 Notation index.

We denote by

$BR(\mathbb{R},X)$	:	space of bounded regulated functions continuous on the left from $\mathbb{R}$ into X
$BC(\mathbb{R},X)$	:	space of bounded continuous functions from $\mathbb{R}$ into X
$BUC(\mathbb{R},X)$	:	space of bounded uniformly continuous functions from $\mathbb{R}$ into X
$R(\lambda,A)$	:	resolvent of A in λ
$D(A)$	:	domain of A
$\overline{D(A)}$	:	closure of $D(A)$
$A_{\vert Y}$	:	part of A in Y
$(t_i)_{i\in\mathbb{Z}}$	:	increasing family of real numbers, support of the impulses
δ	:	$\inf\limits_{i\in\mathbb{Z}}(t_{i+1}-t_i)$
$(T(t))_{t\geq 0}$	:	semigroup of linear operators with impulses
$(S(t))_{t\geq 0}$	:	integrated semigroup of linear operators associated with $(T(t))_{t\geq 0}$

3 Extrapolation space and integrated semigroup

The solution of a non-autonomous linear Cauchy problem on a Banach space X is given, under appropriate conditions, by an evolution family, namely, a family $(U(t,s))_{t\geq s}$ of linear bounded operators on a Banach space X ($U\in\mathcal{L}(X)$), for which the following properties hold :
(i) $U(t,r)U(r,s)=U(t,s)$, for all $t\geq r\geq s\in\mathbb{R}$, $U(t,t)=Id_X$;
(ii) the map $(t,s)\mapsto U(t,s)$ from $\widetilde{D}:=\left\{(t,s)\in\mathbb{R}^2 \mid t\geq s\right\}$ into $\mathcal{L}(X)$ is strongly continuous;
(iii) $\|U(t,s)\|\leq Me^{\omega(t-s)}$ for some $M\geq 1$, $\omega\in\mathbb{R}$ and all $t\geq s$.
A family $(U(t,s))_{t\geq s}$ in $\mathcal{L}(X)$ satisfying (i)-(iii) is called an evolution family (see e.g. [11] [14] [33] [41]).
To an evolution family $U(t,s)_{t\geq s}$, it is useful to associate a semigroups of operators on a Banach space of functions (see e.g. [13] [17] [19] [20] [25] [29] [34] [35] [37] [39] [36]). Denote with $\mathcal{B}$ the Borel algebra of subsets of $\mathbb{R}$, λ the Lebesgue measure on $\mathbb{R}$, and $\mathcal{X}$ a Banach space of real-valued Borel-measurable functions on $\mathbb{R}$ -for example $L^1(\mathbb{R})$- (over $(\mathbb{R},\mathcal{B},\lambda)$).
We set

$$\mathcal{F}(\mathbb{R},X)=\{f:\mathbb{R}\to X \mid f \text{ is strongly measurable and } \|f(.)\|_X\in\mathcal{X}\},$$

then $\mathcal{F}(\mathbb{R},X)$ is a vector space, and for the norm

$$\|f\|_{\mathcal{F}(\mathbb{R},X)}=\|\|f(.)\|_X\|_{\mathcal{X}},\ f\in\mathcal{F}(\mathbb{R},X),$$

$\mathcal{F}(\mathbb{R},X)$ is a Banach space.

Definition 3.1 [30]To every evolution family $U(t,s)_{t\geq s}$ on the Banach space X, we associate the following family of operators on $\mathcal{F}(\mathbb{R},X)$:

$$(T(t)f)(s):=U(s,s-t)f(s-t), \tag{3.1}$$

for $f \in \mathcal{F}(\mathbb{R}, X)$, $s \in \mathbb{R}$ and $t \geq 0$.

We call $\mathcal{F}(\mathbb{R}, X)$ the extrapolation space and $(T(t))_{t\geq 0}$ the extrapolated semigroup. Notice that the translation is a positive, thus bounded, operator on the Banach lattice $\mathcal{X}$, and we have the following lemma (see [40] in II.5.3 and [36]).

Lemma 3.2 $(T(t))_{t\geq 0}$ *defined by (3.1) is an algebraic semigroup of bounded linear operators on* $\mathcal{F}(\mathbb{R}, X)$.

If $(T(t))_{t\geq 0}$ defined by (3.1) is strongly continuous (see details in [24] [30] [26] [28] [27] [38]...), then we can define a Hille-Yosida operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{F}(\mathbb{R}, X)$ associated to $(T(t))_{t\geq 0}$, with constants $M \geq 1$ and $\omega \in \mathbb{R}$, i.e $\mathcal{A}$ is linear, (ω, ∞) is contained in the resolvent set $\rho(\mathcal{A})$ of A and

$$\sup\{\|(\lambda-\omega)^n R(\lambda, \mathcal{A})^n\| : \lambda > \omega;\ n \in \mathbb{N}\} < M, \textbf{(HY)}$$

where $R(\lambda, \mathcal{A}) := (\lambda I - \mathcal{A})^{-1}$ is the resolvent operator of $\mathcal{A}$ at λ. The following result is well-known

Lemma 3.3 *(*[16]*, Theorem 12.2.4)*
The part $(\mathcal{A}_0, D(\mathcal{A}_0))$ *of* $\mathcal{A}$ *in* $X_0 := \overline{D(\mathcal{A})}$ *given by*

$$\mathcal{A}_0 x := \mathcal{A}x,\ D(\mathcal{A}_0) := \{x \in D(\mathcal{A}) : \mathcal{A}x \in X_0\}$$

generates a C_0*-semigroup* $(T_0(t))$ *on* X_0. *Moreover,* $\rho(\mathcal{A}) \subseteq \rho(\mathcal{A}_0)$ *and*

$$R(\lambda, \mathcal{A}_0) = R(\lambda, \mathcal{A})_{|X_0} \text{ for } \lambda \in \rho(\mathcal{A}).$$

The following definitions can be found in Arendt [5].

Definition 3.4 Let E be a Banach space. An integrated semigroup $(S(t))_{t\geq 0}$ is a family of bounded linear operators $S(t)$ on E, with the following properties:
(i) $S(0) = 0$;
(ii) for any $y \in E$, $t \to S(t)y$ is strongly continuous with values in E;
(iii) $S(t)S(s) = \int_0^s S(r+t)dr - \int_0^s S(r)dr$, for all $t, s \geq 0$.

Theorem 3.5 *An operator* $\mathcal{A}$ *is called the generator of an integrated semigroup, if there exists* $\omega \in \mathbb{R}$ *such that* $(\omega, +\infty) \subset \rho(\mathcal{A})$, *and there exists a strongly continuous exponentially bounded family* $(S(t))_{t\geq 0}$ *of linear bounded operators such that* $S(0) = 0$ *and* $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ *for all* $\lambda > \omega$.

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time), that is to say:

Definition 3.6 [7] An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$, there exists a constant $k(\tau) \geq 0$ such that

$$\|S(t) - S(s)\| \leq k(\tau)\,|t-s|\,,\ \text{for all } t, s \in [0, \tau]\,.$$

Theorem 3.7 [18] *Assertions **(i)** and **(ii)** are equivalent :*
(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*
(ii) *A satisfies the condition (**HY**).*

4 semigroup associated to a nonautonomous ordinary differential equation

In this section, we recall the construction of an extrapolation space introduced by Da Prato-Grisvard [10] and Nagel [21], which makes it possible to go from a nonautonomous equation to an autonomous one. We will point out the construction and properties of the semigroup $(T(t))_{t\geq 0}$ associated with an evolution family arising from a nonautonomous ordinary differential equation. This notion will be useful in the sequel.

We now consider a nonautonomous Cauchy problem in a Banach space X

$$\begin{cases} \dfrac{d}{dt}u(t) & = & A(t)u(t) \\ u(s) & = & x \in X \end{cases} \tag{4.1}$$

for $t \geq s \in \mathbb{R}$. $A(t)$ is assumed to be a bounded linear operator, such that for $t \to A(t)$ is continuous, from $\mathbb{R}$ into $\mathcal{L}(X)$.
We denote

$$BUC(\mathbb{R}, X) = \{f : \mathbb{R} \to X : f \text{ is uniformly continuous and bounded.}\},$$

with the norm

$$\|f\| = \sup_{x\in\mathbb{R}} |f(x)| \, .$$

We consider the operator $\mathcal{A}$ on $UBC(\mathbb{R}, X)$ associated with equation (4.1), defined by :

$$(\mathcal{A}f)(s) = -f^{'}(s) + A(s)f(s)$$

with domain

$$D(\mathcal{A}) = \Big\{ f \in BUC(\mathbb{R}, X),\ f \text{ is differentiable and } f^{'} \in BC(\mathbb{R}, X) \Big\} \, .$$

Theorem 4.1 *We suppose that for any t, $A(t)$ is a linear bounded operator, such that $t \to A(t)$ is continuous and uniformly bounded, from $\mathbb{R}$ into $\mathcal{L}(X)$. Then, the operator $\mathcal{A}$ generates a strongly continuous semigroup $T(t)$ in $BUC(\mathbb{R}, X)$.*

Proof : In order to determine the resolvent operator, we must solve the equation

$$\begin{cases} (\lambda I - \mathcal{A})^{-1} f = w \\ w \in D(\mathcal{A}) \end{cases}$$

where $f \in BUC(\mathbb{R}, X)$.
Clearly, w depends on λ. Occasionally, we will use the notation w_λ.
The following formula can be obtained by standard computations

$$w_\lambda(s) = \int_{-\infty}^{s} U(s,t) f(t) e^{\lambda(t-s)} dt,$$

where $(U(s,t))_{s\geq t}$ is an evolution family satisfying

$$\|U(s,t)\| \leq M e^{\omega(s-t)} \text{ for some } M \geq 1, \text{ and } \lambda \geq \omega \in \mathbb{R}.$$

To show the Hille-Yosida property, it is necessary here to consider the n^{th} iterates of $(\lambda I - \mathcal{A})^{-1}$.
We can show that

$$[(\lambda I - \mathcal{A})^{-n} f](s) = \int_{-\infty}^{s} U(s,\sigma) f(\sigma) e^{\lambda(\sigma - s)} \frac{(s-\sigma)^n}{n!} d\sigma,$$

from which we deduce, for $\lambda > \omega$

$$\left\|(\lambda I - \mathcal{A})^{-n} f\right\| \leq \frac{M}{(\lambda - \omega)^n} \|f\| .$$

To conclude the proof of theorem 4.1, it remains to be proven that $D(\mathcal{A})$ is dense in $BUC(\mathbb{R}, X)$. But, $D(\mathcal{A})$ contains obviously $C_b^1(\mathbb{R}, X)$, the space of differentiable functions which are bounded from $\mathbb{R}$ into X, (since any function in $BUC(\mathbb{R}, X)$ is transformed into such an element by convolution with a function in $\mathcal{D}(\mathbb{R}, X)$, the space of functions infinitely many times differentiable from $\mathbb{R}$ into X with bounded support), and this set is obviously dense in $BUC(\mathbb{R}, X)$.$\square$

5 Ordinary differential equation with impulses

We first consider an ordinary differential equation with impulses

$$\begin{cases} \dfrac{du}{dt} = Au(t),\ t > \sigma,\ t \neq t_i,\ i \in \mathbb{Z},\ \sigma \in \mathbb{R}\ (DE) \\ u(\sigma) = \xi \in X,\ (IC) \\ u(t_i^+) = B_i u(t_i^-),\ u(t_i^-) = u(t_i),\ t_i \geq \sigma,\ i \in \mathbb{Z},\ (IMC) \end{cases} \tag{5.1}$$

where

(H_1)-A is a bounded linear operator,

(H_2)-$(B_i)_{i\in\mathbb{Z}}$ is a family of uniformly bounded linear operators, ($\|B_i\| \leq M$, $\forall i$, M is a constant)

(H_3)-$(t_i)_{i\in\mathbb{Z}}$ is an increasing family of real numbers, and there exist $\delta > 0$ and $T < \infty$, such that for any $i \in \mathbb{Z}$,

$$0 < \delta \leq t_{i+1} - t_i \leq T < \infty. \tag{5.2}$$

We first introduce the index function $i(\sigma) = \min\{j : t_j \geq \sigma\}$ where, for each σ, the impulse condition reads in terms of $i(\sigma)$ as

$$u(t_i^+) = B_i u(t_i),\ i \geq i(\sigma)$$

If $\sigma = t_{i(\sigma)}$, that is to say, if we start from an impulse time point, then we use $u(\sigma)$ for $u(\sigma^-)$.

We denote

$$BR(\mathbb{R}, X) = \left\{ \begin{array}{l} f : \mathbb{R} \to X,\ f \text{ is regulated, continuous to} \\ \text{the left and bounded in } \mathbb{R}. \end{array} \right\}. \tag{5.3}$$

We consider the family $U(t,s)_{t\geq s}$ associated to (5.1) defined as follows:

$$U(t,s) = \begin{cases} e^{(t-s)A} & \text{if} \quad [s,t[\cap D = \emptyset, \\ e^{(t-t_i)A} \circ B_i \circ e^{(t_i - s)A} & \text{if} \quad [s,t[\cap D = \{t_i\}, \\ Id & \text{if} \quad t = s, \end{cases} \tag{5.4}$$

Remark 1: $U(t,s)$ is not fully defined by (5.4). Extension of $U(t,s)$ over the whole set W is obtained by using the chain rule property. For arbitrary $t, s,\ t \geq s$ we define $U(t,s)$ as a finite product of operators $U(\tau_{j+1}, \tau_j)$, where $s = \tau_0 < \tau_1 < \ldots < \tau_j < \tau_{j+1} < \ldots < \tau_{N+1} = t$ and $U(\tau_{j+1}, \tau_j)$ is given by formula (5.4), for each j, $0 \leq j \leq N$.

The following lemmas will be needed to proof Theorem 1.1 in the next subsection, and Theorem 1.2 in subsection 5.2.

Lemma 5.1 *The evolution family $U(t,s)$, for all $t \geq s$ associated to (5.1), can be represented by the product of all $U(\tau_{j+1},\tau_j)$, $\tau_j < \tau_{j+1}$, $0 \leq j \leq N$, $(p,q) \in \mathbb{Z}^2$, and we have:*
♦*If* $[s,t[\cap D = \emptyset$,

$$U(t,s) = e^{(t-s)A} \tag{5.5}$$

♦*If* $[s,t[\cap D = \{t_p\}$,

$$U(t,s) = e^{(t-t_p)A} \circ B_p \circ e^{(t_p-s)A} \tag{5.6}$$

♦*If* $[s,t[\cap D = \{t_i, i = p, p+1, ..., q; q > p,\ (p,q) \in \mathbb{Z}^2\}$,

$$U(t,s) = e^{(t-t_q)A} \circ B_q \circ e^{(t_q-t_{q-1})A} \circ ... \circ B_p \circ e^{(t_p-s)A}. \tag{5.7}$$

Proof: We consider the general case when $[s,t[\cap D$ has $q-p+1$ elements with $q > p$, then we have

$$[s,t[\cap D = \{t_i, i = p, p+1, ..., q; q > p, (p,q) \in \mathbb{Z}^2\},$$

and if we consider a finite family $(\tau_l)_{0\leq l\leq N+1}$, such that $\tau_l < \tau_{l+1}$ $s = \tau_0$, $\tau_{N+1} = t$. In order for $U(\tau_{l+1},\tau_l)$ to be defined by (5.4), it is necessary that if for some l and p we have: $t_p \in [\tau_l,\tau_{l+1}[$, then, for this l, $[\tau_l,\tau_{l+1}[\cap D = \{t_p\}$. So, in view of (5.4), we will have:
if $[\tau_l,\tau_{l+1}[\cap D = \emptyset$

$$U(\tau_{l+1},\tau_l) = e^{(\tau_{l+1}-\tau_l)A},$$

or, if $[\tau_l,\tau_{l+1}[\cap D = \{t_p\}$

$$U(\tau_{l+1},\tau_l) = e^{(\tau_{l+1}-t_p)A} \circ B_p \circ e^{(t_p-\tau_l)A}.$$

Taking the product of $U(\tau_{l+2},\tau_{l+1})$ and $U(\tau_{l+2},\tau_{l+1})$, we obtain:
if $[\tau_l,\tau_{l+2}[\cap D = \{t_p\}$

$$U(\tau_{l+2},\tau_{l+1}) = e^{(\tau_{l+2}-\tau_{l+1})A},$$

or
if $[\tau_l,\tau_{l+2}[\cap D = \{t_p, t_{p+1}\}$

$$U(\tau_{l+2},\tau_l) = e^{(\tau_{l+2}-t_{p+1})A} \circ B_{p+1} \circ e^{(t_{p+1}-t_p)A} \circ B_p \circ e^{(t_p-\tau_l)A}.$$

We can similarly represent the product of all the $(U(\tau_{l+1},\tau_l))_{0\leq l\leq N}$. We arrive at

$$U(t,s) = e^{(t-t_q)A} \circ B_q \circ e^{(t_p-t_{p-1})A} \circ ... \circ B_q \circ e^{(t_q-s)A}.$$

Obviously, this expressions independent on the family $(\tau_l)_{0\leq l\leq N+1}$, and we obtained doubly indexed family of operators $U(t,s)$ satisfies the chain rule property

$$U(t,s) = U(t,r) \circ U(r,s),$$

for all t,r,s such that $t \geq r \geq s$.□

We note in this section, that the limit is not uniform but, is only a **pointwise limit.**

Lemma 5.2 *Consider equation (5.1) with $(B_i)_{i\in\mathbb{Z}}$ and D satisfying assumptions (H_2) and (H_3), and $U(t,s)$ given as said above (5.5)-(5.7). Then, $t \to U(t,s)$ is continuous at any point $t \notin D$, and continuous to the left at any t_i, and we have*

$$\lim_{\substack{t\to t_i\\ t>t_i}} U(t,s) = B_i \lim_{\substack{t\to t_i\\ t<t_i}} U(t,s) = B_i U(t_i,s). \tag{5.8}$$

Proof: Let $U(t,s)$ be the evolution family associated to (5.1), and suppose first that $s,t\in\mathbb{R}$, be such that $0<t-s<\delta$. Then :
•If $s<t$ and $t_i\notin[s,t[$ then : $U(t,s)=\exp((t-s)A)$. Consequently, $t\to U(t,s)$ is continuous.
•If $t_i\in[s,t[$, in view of (5.2), it is the only point of the set $\{t_i : i\in Z\}$. Then

$$U(t,s)=\exp((t-t_i)A)\circ B_i\circ\exp((t_i-s)A).$$

Thus,

$$\lim_{\substack{t\to t_i\\ t>t_i}} U(t,s)=B_i\lim_{\substack{t\to t_i\\ t<t_i}} U(t,s)=B_iU(t_i,s).$$

In order to extend the property to an arbitrary pair (t,s), $t>s$, we just have to express $U(t,s)$ a product

$$U(t,s)=U(t,\tau^*)\circ U(\tau^*,s)$$

where we assume that $0<t-\tau^*<\delta$. Then, the first step yields continuity of $t\to U(t,\tau^*)$ at any $t\notin D$, and

$$\lim_{\substack{t\to t_i\\ t>t_i}} U(t,\tau^*)=B_i\lim_{\substack{t\to t_i\\ t<t_i}} U(t,\tau^*)=B_iU(t_i,\tau^*).$$

Taking the product of the above with $U(\tau^*,s)$, formula (5.8) extends to any pair (t,s), $t>s$.$\square$

5.1 Pointwise regulated semigroup. We give the following definition.

Definition 5.3 Let X be a Banach space. A one parameter family $(T(t))_{t\geq 0}$, of bounded linear operators on $BR(\mathbb{R},X)$ is a pointwise regulated semigroup if :
(i) $(T(t))_{t\geq 0}$ is an algebraic semigroup.
(ii) for any fixed $f\in BR(\mathbb{R},X)$, $s\in\mathbb{R}$ and $t\geq 0$, both the maps :

$$\text{and}\quad \left.\begin{array}{l} t\to (T(t)f)(s)\\ s\to (T(t)f)(s)\end{array}\right\}\ \text{are regulated.}$$

We will now show that the semigroup $(T(t))_{t\geq 0}$ defined by (3.1) associated with the evolution family $U(t,s)$, defined by formula (5.5)-(5.7) in lemma 5.1, is pointwise regulated. We have $(T(t))_{t\geq 0}$ defined, for any $f\in BR(\mathbb{R},X)$, $s\in\mathbb{R}$, $t\geq 0$, $(n,m)\in\mathbb{Z}^2$ as follows :

From the expression (1.5) and $(H_1)-(H_3)$, we have the following estimation:

$$\begin{array}{rcl} \|T(t)f\| &=& \sup\limits_s |(T(t)f)(s)|\\ &\leq& e^{t\|A\|}\sup\limits_s(\prod\limits_{s-t<t_i<s}\|B_i\|)\sup\limits_s(\sup\limits_{s-t<\tau<s}|f(\tau)|)\\ \|T(t)f\| &\leq& M_te^{t\|A\|}\,\|f\| \end{array}\tag{5.9}$$

where $M_t=\sup\limits_s(\prod\limits_{s-t<t_i<s}\|B_i\|)$, $\|f\|=\sup\limits_s(\sup\limits_{s-t<\tau<s}|f(\tau)|)$.
We are now in position to prove the first main result of this work, namely, Theorem 1.1 (stated in the Introduction):
Proof of Theorem 1.1: We first point out that -due to semigroup properties- $(T(t))_{t\geq 0}$ being pointwise regulated is equivalent to $(T(t)f)(s)$ being separately regulated in t and s, for $0\leq t<\tau$, and all $s\in\mathbb{R}$, for some $\tau>0$.

So, we will assume for the rest of the proof that $0 \leq t < \delta$ (where δ is as in (H$_3$)).
For $f \in BR(\mathbb{R}, X)$, we have

$$(T(t)f)(s) = U(s, s-t)[f(s-t)],\ \forall t \geq 0.$$

•We fix $t \geq 0$, then we have for : $t_i - \delta < s \leq t_i$,

$$(T(t)f)(s) = e^{tA} f(s-t).$$

In view of (H_1), then $s \to (T(t)f)(s)$ is regulated.
For : $s - t \leq t_i < s$, we have

$$(T(t)f)(s) = e^{(s-t_i)A} \circ B_i \circ e^{(t_i-(s-t))A} f(s-t).$$

From this expression we can using (H_1) and (H_2) that $s \to (T(t)f)(s)$ is regulated at every $s \neq t_i$. We now consider the situation at $s = t_i$

$$\lim_{\substack{s \to t_i \\ \leq}} (T(t)f)(s) = e^{tA} \lim_{\substack{s \to t_i \\ \leq}} f(s-t)$$

and

$$\lim_{\substack{s \to t_i \\ >}} (T(t)f)(s) = B_i e^{tA} \lim_{\substack{s \to t_i \\ >}} f(s-t).$$

Therefore, $s \to (T(t)f)(s)$ is regulated over $\mathbb{R}$.
•In the same way, for fixed s, we have

$$(T(t)f)(s) = \begin{cases} e^{At} f(s-t), & \text{if} \quad [s-t, s[\cap D = \emptyset, \\ e^{(s-t_i)A} \circ B_i \circ e^{(t_i-(s-t))A} f(s-t) & \text{if} \quad [s-t, s[\cap D = \{t_i\}. \end{cases}$$

Then, $t \to (T(t)f)(s)$ is regulated, for all $0 \leq t < \delta$.
By using propriety of algebraic semigroup, for fixed $t \in [0, \delta]$, and for small h, we have

$$(T(t+h)f)(s) = (T(t)T(h)f)(s), \text{ for } f \in BR(\mathbb{R}, X),$$

since, from the first step yields that, for small h, $T(h)$ is a pointwise regulated semigroup of bounded linear operators on $BR(\mathbb{R}, X)$.
Then, we have the result for all $t \geq 0$.

Remark 2: The map $(t, s) \to (T(t)f)(s)$ is continuous at any point (t, s), $t \geq 0$, $s \in \mathbb{R}$, such that neither s nor $s - t$ is in the set $D_f \cup \{t_i\}$, where D_f is the set of the points of discontinuity of f. Then, $(t, s) \to (T(t)f)(s)$ is continuous outside of a negligible set in $\mathbb{R}^+ \times \mathbb{R}$.

5.2 Integrated semigroup. We showed in Theorem 1.1 that the map $t \to (T(t)f)(s)$ is regulated for every s. As a consequence, we can integrate the function in t for, each fixed s. This yields an operator $S(t)$ on $BR(\mathbb{R}, X)$ given by

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s) d\tau,$$

for $t \geq 0$, $f \in BR(\mathbb{R}, X)$, and $T(t)$ given by (1.3) or (1.5) for all $s \in \mathbb{R}$.
$S(t)f$ is well defined by formula (1.6). We note that this expression is not defined as a vector integral on $BR(\mathbb{R}, X)$ but as a function defined at each point s as the integral of a regulated function. We will now prove the Theorem 1.2 (stated in the Introduction):

Proof of Theorem 1.2: Let $f \in BR(\mathbb{R}, X)$, then for small h, we have for any $t \in \mathbb{R}^+$

$$S(t+h)f - S(t)f = \int_t^{t+h} T(\tau) f d\tau. \tag{5.10}$$

From (5.9), we have,

$$\|T(\tau)f\| \leq M_{t+1} e^{(t+1)\|A\|} \|f\|, \text{ for any } \tau \in [t, t+h],$$

where $M_t = \sup_s (\prod_{s-t<t_i<s} \|B_i\|)$.

Thus,

$$\|S(t+h)f - S(t)f\| \leq M_{t+1} e^{(t+1)\|A\|} h \|f\|,$$

which shows the norm continuity.

For the algebraic formula, we have for any $f \in BR(\mathbb{R}, X)$, $\eta \in \mathbb{R}$,

$$\begin{aligned}
\int_0^s (S(\tau+\sigma) - S(\sigma))(f)(\eta) d\sigma &= \int_0^s ((S(t+\sigma) - S(\sigma))f)(\eta) d\sigma \\
&= \int_0^s (\int_0^{t+\sigma} (T(\tau)f)(\eta) d\tau - \\
&\quad \int_0^{\sigma} (T(\tau)f)(\eta) d\tau) d\sigma \\
&= \int_0^s (\int_\sigma^0 (T(\tau)f)(\eta) d\tau + \\
&\quad \int_0^{t+\sigma} (T(\tau)f)(\eta) d\tau) d\sigma \\
&= \int_0^s \int_\sigma^{t+\sigma} (T(\tau)f)(\eta) d\tau d\sigma \\
&= \int_0^s \int_0^t (T(\tau+\sigma)f)(\eta) d\tau d\sigma \\
&= \int_0^s \int_0^t (T(\tau)T(\sigma)f)(\eta) d\tau d\sigma \\
&= \int_0^s (T(\tau) \int_0^t T(\sigma)f)(\eta) d\sigma d\tau \\
&= \int_0^s (T(\tau)S(t)f)(\eta) d\tau \\
&= \int_0^s T(\tau)S(t)(f)(\eta) d\tau \\
&= (S(s)S(t)f)(\eta).
\end{aligned}$$

Then

$$S(s)S(t) = \int_0^s (S(t+\sigma) - S(\sigma)) d\sigma,$$

and

$$S(0) = 0.$$

Consequently, $S(t)$ is a norm continuous integrated semigroup on $BR(\mathbb{R}, X)$.□

We are now in position to prove the first main result of this work, namely, Theorem 1.3 (stated in the Introduction):

Proof of Theorem 1.3: Let $f \in BR(\mathbb{R}, X)$. For t fixed such that $0 \leq t < \delta$, we have

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau.$$

There are 2 cases to consider :

a) If $t_i < s < t_i + \delta$ and $s - t \leq t_i$, we have

$$\begin{aligned}(S(t)f)(s) &= \int_0^{s-t_i} (T(\tau)f)(s)d\tau + \int_{s-t_i}^t (T(\tau)f)(s)d\tau \\ &= \int_0^{s-t_i} e^{\tau A} f(s-\tau)d\tau + \\ &\quad \int_{s-t_i}^t e^{(s-t_i)A} \circ B_i \circ e^{(t_i-(s-\tau))A} f(s-t)d\tau.\end{aligned}$$

b) If $s < t_{i+1}$ and $s - t > t_i$, then

$$\begin{aligned}(S(t)f)(s) &= \int_0^t (T(\tau)f)(s)d\tau \\ &= \int_0^t e^{\tau A} f(s-\tau)d\tau.\end{aligned}$$

We notice that :

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s) = \int_0^t B_i \exp(\tau A) \lim_{\substack{s \to t_i \\ >}} f(s-\tau)d\tau$$

and

$$\lim_{\substack{s \to t_i \\ <}} (S(t)f)(s) = \int_0^t \exp(\tau A) \lim_{\substack{s \to t_i \\ <}} f(s-\tau)d\tau.$$

Then, $s \to (S(t)f)(s)$ is continuous at each point $s \notin \{t_i\}_{i \in \mathbb{Z}}$ and thus has possible discontinuities only at points t_k. By (5.10) in Theorem 1.2, we have

$$\begin{aligned}(S(t+h)f)(s) &= (S(t)f)(s) + \int_0^h (T(\tau+t)f)(s)d\tau \\ &= (S(t)f)(s) + \int_0^h (T(\tau)T(t)f)(s)d\tau \\ &= (S(t)f)(s) + (S(h)T(t)f)(s).\end{aligned}$$

By using Theorem 1.1, $t \to T(t)f$ is regulated, for any $f \in BR(\mathbb{R}, X)$.
Then, we have the result for all $t \geq 0$.□

We remark that the integration of the semigroup made it possible to eliminate all discontinuities due to the data, except at points $s = t_i$, $i \in \mathbb{Z}$.

6 Generator and domain of the integrated semigroup

We denote the set singularities of a function f, (i.e the points of discontinuity of f on $\mathbb{R}$) by sing(f). In Theorem 1.4, described in Introduction , we will show that the operator $\mathcal{A}$ defined hereafter is the infinitesimal generator of the continuous integrated semigroup $S(t)$:

$$(\mathcal{A}f)(s) = Af(s) - f'(s)$$

$$D(\mathcal{A}) = \left\{ \begin{array}{l} f \in BR(\mathbb{R}, X) : \ \text{sing} f \subset \{t_i\}_{i\in\mathbb{Z}}, \ \lim\limits_{\substack{s\to t_i \\ >}} f(s) = \lim\limits_{\substack{s\to t_i \\ <}} B_i f(s), \\ \text{and } \dfrac{\partial f}{\partial s} \in BR(\mathbb{R}, X), \ \text{sing}\dfrac{\partial f}{\partial s} \subset \{t_i\}_{i\in\mathbb{Z}}, \end{array} \right\}.$$

Proof of Theorem 1.4: We will show that $\mathcal{A}$ is a Hille-Yosida operator. For this, we first have to determine the resolvent operator:

$$R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}.$$

Given $f \in BR(\mathbb{R}, X)$, the equation reads as (with $v \in D(\mathcal{A})$)

$$\lambda v(s) - \mathcal{A}v(s) = f(s). \tag{6.1}$$

Then,

$$\lambda v(s) - Av(s) + v^{'}(s) = f(s),$$

and by integration, we respect to $s \in]t_i, t_{i+1}[$

$$v(s) = \exp((A-\lambda)(s-t_i))v(t_i^+) + \int_{t_i}^{s} \exp((A-\lambda)(s-\sigma))f(\sigma)d\sigma. \tag{6.2}$$

For $i \in \mathbb{Z}$, let $V_i = v(t_i)$, be a sequence determine by

$$V_i = \exp((A-\lambda)(t_i - t_{i-1}))B_{i-1}V_{i-1} + \int_{t_{i-1}}^{t_i} \exp((A-\lambda)(t_i - t_{i-1}))f(\sigma)d\sigma. \tag{6.3}$$

As a Hypothesis $(H_1) - (H_3)$. Then for $\lambda > 0$ large enough, the first expression of equation (6.3) :

$$\Delta_I : V \to (\exp((A-\lambda)(t_i - t_{i-1})) \circ B_{i-1}V_{i-1})_{i\in\mathbb{Z}}$$

is a strict contraction in the space $l_\infty(\mathbb{Z})$ and consequently from (6.3) we have :

$$V = (I - \Delta_I)^{-1}(\int_{t_{i-1}}^{t_i} e^{(A-\lambda)(t_i - t_{i-1})} f(\sigma)d\sigma)_{i\in\mathbb{Z}},$$

and thus

$$\|V\| \leq \frac{e^{(\|A\|-\lambda)(t_i - t_{i-1})}}{1 - \|\Delta_I\|}(t_i - t_{i-1}) \|f\|. \tag{6.4}$$

Consequently for $s \in \mathbb{R}$, and from (6.2),

$$\begin{aligned} \|v\| &\leq e^{(\|A\|-\lambda)(s-t_i)} \|B_i\| \frac{e^{(\|A\|-\lambda)(t_i - t_{i-1})}}{1 - \|\Delta_I\|}(t_i - t_{i-1}) \|f\| \\ &\quad + \|f\| \int_{t_i}^{s} e^{(\|A\|-\lambda)(s-\sigma)} d\sigma \\ &\leq e^{(\|A\|-\lambda)(s-t_i)} \|B_i\| \frac{e^{(\|A\|-\lambda)(t_i - t_{i-1})}}{1 - \|\Delta_I\|}(t_i - t_{i-1}) \|f\| \\ &\quad + \frac{\|f\|}{\lambda - \|A\|}(1 - e^{(\|A\|-\lambda)(s-t_i)}) \\ &\leq e^{(\|A\|-\lambda)(s-t_i)} \left[\|B_i\| \frac{e^{(\|A\|-\lambda)(t_i - t_{i-1})}}{1 - \|\Delta_I\|}(t_i - t_{i-1}) - \frac{1}{\lambda - \|A\|} \right] \|f\| \\ &\quad + \frac{\|f\|}{\lambda - \|A\|}. \end{aligned}$$

Let

$$\Phi(z) = \frac{1}{z} - e^{-z\alpha} \frac{\alpha \|B_i\|}{1 - \|\Delta_I\|}, \text{ with } \alpha, z > 0.$$

From (1.7), we have for α, $z > 0$,

$$\begin{aligned} \Phi(z) &\geq \frac{1}{z} - e^{-z\alpha}\frac{\alpha e}{1-\|\Delta_I\|} \\ &\geq \frac{1-\|\Delta_I\| - \alpha z e^{-z\alpha}\alpha e}{z(1-\|\Delta_I\|)}, \end{aligned}$$

and for $\lambda > 0$, large enough, we have $\|\Delta_I\|$ small, and $\frac{1}{e} \geq \alpha z e^{-z\alpha}$, then

$$\Phi(z) \geq 0.$$

Consequently, for $\lambda > \|A\|$ we have :

$$e^{(\|A\|-\lambda)(t_i - t_{i-1})}\frac{(t_i - t_{i-1})\,\|B_i\|}{1-\|\Delta_I\|} \leq \frac{1}{\lambda - \|A\|}.$$

Finally

$$\|v\| \leq \frac{\|f\|}{\lambda - \|A\|},\ \lambda \geq \|A\|$$

In generally, we have

$$\left|(\lambda I - \mathcal{A})^{-n} f\right| \leq \|f\|_\infty \frac{1}{(\lambda - \|A\|)^n}.$$

Thus, $\mathcal{A}$ is a Hille-Yosida operator, and therefore it determines an locally Lipschitz continuous integrated semigroup $(S(t)f)$ on $BR(\mathbb{R}, X)$.□

From the theory of integrated semigroup (Lemma 3.3 and Theorem 3.5), see the literature [1] [2] [3] [4] [5] [6] [7], it is known that $T(t)_{|\overline{D(\mathcal{A})}}$, $t \geq 0$, constitutes a C_0-semigroup on $\overline{D(\mathcal{A})}$, with the same infinitesimal generator $\mathcal{A}$. For delay differential equation, we can see [9].

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