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# Integrated semigroup associated to a linear delay differential equation with impulses

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**Abstract.** In this paper, we discuss the fundamental linear theory for a large class of delay differential equations with impulses. We show, using the general theory of integrated semigroups, that we can associate a integrated semigroup with any delay differential equation with impulses, and we have determined the infinitesimal generator of this integrated semigroup.

## 1 Introduction

Differential equations with impulses have been considered for the first time by Milman and Myshkis (see [23], [24]). They formalize the situation when the state of a system changes as a result of jumps occurring at different moments of time. The times at which jumps occur may be known and form a sequence of times with or without a certain pattern, or may be determined in terms of the state itself. We remark that the problem with impulses is no more an autonomous problem. Since the early work of Milman and Myshkis, this type of equation has been considered extensively in the literature (see the monographs [18], [33]). In recent years, many examples of differential equations with impulses have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of

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a drug treatment or a missing product. In environmental sciences, seasonal changes of the water level of artificial reservoirs, as well as under the effect of floodings, can be modeled as impulses, see for example [34],[39],[40]. In this paper we consider ordinary and delay differential equations with impulses in any Banach space. By an impulse, we mean a sudden change of the state of a system. At each moment of a possibly unbounded sequence of moments, the state jumps from one position to another, as a consequence of a transformation which depends only on the moment of the impulses. In section 2, we illustrate the method of extrapolation and of integrated semigroup. In section 3, we apply the method of extrapolation for a nonautonomous ordinary differential equation without impulses.
Our purpose is to provide a linear theory for such differential equations with impulses in Banach spaces. There are two main challenges: the first one is set by the jump discontinuities which make it necessary to extend the usual state space of continuous functions to a space of functions having some discontinuities; the second one is the time-dependence of the system, arising implicitly from the time jumps. The strategy used to overcome these two problems is two-fold:
1) Time-dependence will be eliminated by a recurse to extrapolation theory, for the ordinary (respectively delay) differential equations with impulses in subsection 4.1 (respectively subsection 5.1); see, e.g., [15], [9], [25], [26], [32], [28], [30], [29], [42].
2) Integration will be used to smooth down the discontinuities, for the ordinary (respectively delay) differential equations with impulses in subsection 4.2 (respectively subsection 5.2). This goes through the now well-known integrated semigroup theory; see, e.g., [1], [2], [3], [4], [5], [6], [7].
The prototype equation reads as follows:

$$\begin{cases} \dfrac{du}{dt} = Lu_t,\ t > \sigma,\ t \neq t_i,\ i \in \mathbb{Z},\ \sigma \in \mathbb{R} \\ u_0(t) = \phi(t-\sigma),\ \sigma - r \leq t \leq \sigma, \\ u(t_i^+) = B_i u(t_i^-),\ u(t_i^-) = u(t_i),\ t_i \geq \sigma,\ i \in \mathbb{Z} \end{cases} \tag{1}$$

$X$ is any Banach space. The operator $L$ is initially defined from $C([-r,0],X)$ into $X$; the last equation introduces the jumps, which make it necessary to extend $L$ to a slightly larger class of initial values, namely, a natural space in the context of impulses is the space of regulated $X$- valued functions on $[-r,0]$-, that is to say, functions which have limits both to the right (with the exception of 0) and to the left (with the exception of $-r$). In general, we say that the function $f$ is regulated if $f$ has left and right limits at every point (the limit here is not uniform but only a pointwise limit), and we denote by $\lim\limits_{<}$ (respectively $\lim\limits_{>}$), the **pointwise limit** to the right (respectively left), see, e.g., [8], [12] .
Throughout the paper, we denote by $D = \{t_i, i \in \mathbb{Z}\}$ the set of times of impulses, and we will use the notation $R([-r,0],X) = R_X$, for the space of regulated functions, and $R_X^+ = R_X \times X$ will denote the set of pairs $(\phi, x)$ made up of an element $\phi$ of $R_X$ and an element $x$ in $X$ which stands for the limit to the right at 0 of a potential extension of $\phi$. We will also consider a subspace of $R_X$, namely, the functions which are continuous to the left at each point of the interval $]-r,0]$. From now on, the notation $R_X$ stands for the space of regulated functions, continuous to the left. On occasion, it will be convenient to consider $R_X$ as a subset of $R_X^+$, that is,

$$R_X = \left\{(\phi, x) \in R_X^+ : x = \phi(0)\right\}. \tag{2}$$

We will now describe the main results and the main steps to be accomplished in order to derive these results. Throughout the paper, we denote $U(t,s)$ the evolution operator which maps initial values, given at time $s$, to the solution at any future time $t$, and $T(t)$ the operator defined as following

$$(T(t)f)(s) = U(s, s-t)(f(s-t)), \tag{3}$$

where $f \in BR(\mathbb{R}, X)$ (respectively $f \in BR(\mathbb{R}, R_X^+)$) for ordinary (respectively delay) differential equations with impulses, $s \in \mathbb{R}$, $t \geq 0$, where $BR(\mathbb{R}, X)$ (respectively $BR(\mathbb{R}, R_X^+)$) is the space of bounded regulated functions $\mathbb{R} \to X$ (respectively $\mathbb{R} \to R_X$) continuous to the left.
In subsection 4.1 we associate the operator $T(t)$ to the ordinary differential equation with impulses by using formula (3) and the evolution operator $U(\cdot,\cdot)$ associated with the linear ordinary differential equations with impulses.
For the delay differential equation without impulses in (1), we associate the semigroup $T_L$ on $BR(\mathbb{R}, R_X^+)$. Similar to ordinary differential equations with impulses, the operator $T(t)$ defined by formula (3) associated with delay differential equations with impulses (1), in subsection 5.1, can be writhed as:
•If $[s-t, s[ \cap D = \emptyset$,

$$(T(t)f)(s) = T_L(t)[f(s-t)] \tag{4}$$

•If $[s-t, s[ \cap D = \{t_n\}, n \in \mathbb{Z}$

$$(T(t)f)(s) = T_L(s-t_n) \circ \widetilde{B}_n \circ T_L(t_n - (s-t))[f(s-t)] \tag{5}$$

•If $[s-t, s[ \cap D = \{t_i, i = n, n+1, ..., m; m > n, (n,m) \in \mathbb{Z}^2\}$,

$$(T(t)f)(s) = T_L(s-t_m) \circ \widetilde{B}_m \circ T_L(t_m - t_{m-1}) \circ ... \circ \widetilde{B}_n \circ T_L(t_n - (s-t))[f(s-t)] \tag{6}$$

where $\widetilde{B}_i$ maps $\phi \in R_X$ to the function $\widetilde{B}_i(\phi) = (\phi, B_i(\phi(0))) \in R_X^+$.
Our first result states that, the operator $(T(t))_{t\geq 0}$ for ordinary (respectively delay) differential equations with impulses, in subsection 4.1 (respectively in subsection 5.1), is a pointwise regulated semigroup of bounded linear operators on $BR(\mathbb{R}, X)$ (respectively $BR(\mathbb{R}, R_X^+)$). We introduce the following family of operators

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau, \ t \geq 0, \tag{7}$$

for $f \in BR(\mathbb{R}, X)$ (respectively $f \in BR(\mathbb{R}, R_X^+)$) for the ordinary (respectively delay) differential equations with impulses, $s \in \mathbb{R}$, $t \geq 0$, and we show that, the operator $(S(t))_{t>0}$ defined by formula (7), for ordinary case in subsection 4.2 (respectively delay case in subsection 5.2), is a norm continuous integrated semigroup on $BR(\mathbb{R}, X)$, (respectively $BR(\mathbb{R}, R_X^+)$).

For delay differential equations with impulses, we have proven the following results, also valid, of course, for the ordinary differential equations with impulses, see subsection 4.2.

**Theorem 1** *We assume that $S$ is the integrated semigroup associated with equation (1). Let $f \in BR(\mathbb{R}, R_X^+)$, then the map $s \to (S(t)f)(s)(\theta)$, $(t \geq 0, \theta \in [-r, 0])$ is continuous at each $s \notin \{t_i\}$, and for all $t > 0$ fixed, we have*

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s)(0) = B_i \left[ \lim_{\substack{s \to t_i \\ <}} (S(t)f)(s)(0) \right].$$

An important feature revealed by this theorem is the fact that the integrated semigroup takes its values in the space of functions whose discontinuities are concentrated in the set $D$, the set of times of jumps. This weak regularizing property is the analog of what happens in integrated semigroups.

Finally, in section 6, we describe the infinitesimal generator associated with the semigroup $T(t)$:

**Theorem 2** *The operator $\mathcal{A}$ defined by*

$$D(\mathcal{A}) = \left\{ \begin{array}{l} f \in BR(\mathbb{R}, R_X^+) : \ singf \subset \{t_i\}_{i\in\mathbb{Z}}, \ \lim\limits_{\substack{s \to t_i \\ >}} f(s) = \lim\limits_{\substack{s\to t_i \\ <}} \widetilde{B}_i f(s), \\ and \ \dfrac{\partial f(s)}{\partial \theta}(\theta) \in R_X^+, \ \dfrac{\partial f}{\partial s} \in BR(\mathbb{R}, R_X^+), \ sing\dfrac{\partial f}{\partial s} \subset \{t_i\}_{i\in\mathbb{Z}}, \\ \dfrac{\partial f}{\partial \theta}(s)(0) = Lf(s), \ s \notin \{t_i\}_{i\in\mathbb{Z}}. \end{array} \right\}.$$

*and*

$$\begin{aligned} (\mathcal{A}f)(s) &= \frac{\partial}{\partial\theta} f(s)(\theta) - \frac{\partial}{\partial s} f(s)(\theta), \ s \notin \{t_i\} \\ (\mathcal{A}f)(s)(0) &= \frac{\partial f}{\partial \theta}(s) - f^{'}(s)(0) \end{aligned}$$

*is the generator of locally Lipschitz continuous integrated semigroup $S(t)$ on $BR(\mathbb{R}, R_X^+)$, which satisfies $S(t)(BR(\mathbb{R}, R_X^+)) \subset C(\mathbb{R} - \{t_i\}_{i\in\mathbb{Z}}, R_X^+)$ and*

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s) d\tau, \ for \ f \in BR(\mathbb{R}, R_X^+),$$

*where $T(t)$ is defined by (4)-(6).*

From the theory of integrated semigroups, it is known that $T(t)_{|\overline{D(\mathcal{A})}}$, $t \geq 0$, constitutes a $C_0$-semigroup on $\overline{D(\mathcal{A})}$, with the same infinitesimal generator $\mathcal{A}$. See the literature [1], [2], [3], [4], [5], [6], [7].

## 2 Extrapolation space and integrated semigroup

The solution of a non-autonomous linear Cauchy problem on a Banach space $X$ is given, under appropriate conditions, by an evolution family, namely, a family $(U(t,s))_{t\geq s}$ of linear bounded operators on a Banach space $X$ ($U \in \mathcal{L}(X)$), for which the following properties hold :
(i) $U(t,r)U(r,s) = U(t,s)$, for all $t \geq r \geq s \in \mathbb{R}$, $U(t,t) = Id_X$;
(ii) the map $(t,s) \mapsto U(t,s)$ from $\widetilde{D} := \left\{(t,s) \in \mathbb{R}^2 \mid t \geq s\right\}$ into $\mathcal{L}(X)$ is strongly continuous;
(iii) $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for some $M \geq 1$, $\omega \in \mathbb{R}$ and all $t \geq s$.
A family $(U(t,s))_{t\geq s}$ in $\mathcal{L}(X)$ satisfying (i)-(iii) is called an evolution family (see e.g., [10], [17], [35], [45]).
To an evolution family $U(t,s)_{t\geq s}$, it is useful to associate a semigroup of operators on a Banach space of functions (see e.g., [16], [20], [22],[15], [27], [31], [36], [37], [41], [43], [38]). Denote with $\mathcal{B}$ the Borel algebra of subsets of $\mathbb{R}$, $\lambda$ the Lebesgue measure on $\mathbb{R}$, and $\mathcal{X}$ a Banach space of real-valued Borel-measurable functions on $\mathbb{R}$ -for example $L^1(\mathbb{R})$- (over $(\mathbb{R}, \mathcal{B}, \lambda)$).
We set

$$\mathcal{F}(\mathbb{R}, X) = \{f : \mathbb{R} \to X \mid f \text{ is strongly measurable and } \|f(.)\|_X \in \mathcal{X}\},$$

then $\mathcal{F}(\mathbb{R},X)$ is a vector space, and for the norm

$$\|f\|_{\mathcal{F}(\mathbb{R},X)} = \left\| \|f(.)\|_X \right\|_{\mathcal{X}}, \ f \in \mathcal{F}(\mathbb{R},X),$$

$\mathcal{F}(\mathbb{R},X)$ is a Banach space.

**Definition 3** [32] *To every evolution family $(U(t,s))_{t\geq s}$ on the Banach space $X$, we associate the following family of operators on $\mathcal{F}(\mathbb{R},X)$:*

$$(T(t)f)(s) := U(s,s-t)f(s-t), \tag{8}$$

*for $f \in \mathcal{F}(\mathbb{R},X)$, $s \in \mathbb{R}$ and $t \geq 0$.*

We call $\mathcal{F}(\mathbb{R},X)$ the extrapolation space and $(T(t))_{t\geq 0}$ the extrapolated semigroup. Notice that the translation is a positive, thus bounded, operator on the Banach lattice $\mathcal{X}$, and we have the following lemma ( see [44] in II.5.3 and [38]).

**Lemma 4** *$(T(t))_{t\geq 0}$ defined by (8) is an algebraic semigroup of bounded linear operators on $\mathcal{F}(\mathbb{R},X)$.*

If $(T(t))_{t\geq 0}$ defined by (8) is strongly continuous ( see details in [15], [26], [32], [28], [30], [29], [42]), then we can define a Hille-Yosida operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{F}(\mathbb{R},X)$ associated to $(T(t))_{t\geq 0}$, with constants $M \geq 1$ and $\omega \in \mathbb{R}$, i.e $\mathcal{A}$ is linear, $(\omega,\infty)$ is contained in the resolvent set $\rho(\mathcal{A})$ of $A$ and

$$\sup\left\{\|(\lambda-\omega)^n R(\lambda,\mathcal{A})^n\| : \lambda > \omega;\ n \in \mathbb{N}\right\} < M, \quad \textbf{(HY)}$$

where $R(\lambda,\mathcal{A}) := (\lambda I - \mathcal{A})^{-1}$ is the resolvent operator of $\mathcal{A}$ at $\lambda$. The following result is well-known

**Lemma 5** *([19], Theorem 12.2.4)*
*The part $(\mathcal{A}_0, D(\mathcal{A}_0))$ of $\mathcal{A}$ in $X_0 := \overline{D(\mathcal{A})}$ given by*

$$\mathcal{A}_0 x := \mathcal{A}x,\ D(\mathcal{A}_0) := \{x \in D(\mathcal{A}) : \mathcal{A}x \in X_0\}$$

*generates a $C_0$-semigroup $(T_0(t))$ on $X_0$. Moreover, $\rho(\mathcal{A}) \subseteq \rho(\mathcal{A}_0)$ and*

$$R(\lambda,\mathcal{A}_0) = R(\lambda,\mathcal{A})_{|X_0} \text{ for } \lambda \in \rho(\mathcal{A}).$$

The following definitions can be found in Arendt [5].

**Definition 6** *Let $E$ be a Banach space. An integrated semigroup $(S(t))_{t\geq 0}$ is a family of bounded linear operators $S(t)$ on $E$, with the following properties:*
*(i) $S(0) = 0$;*
*(ii) for any $y \in E$, $t \to S(t)y$ is strongly continuous with values in $E$;*
*(iii) $S(t)S(s) = \int_0^s S(r+t)dr - \int_0^s S(r)dr$, for all $t,s \geq 0$.*

**Theorem 7** *An operator $\mathcal{A}$ is called the generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega,+\infty) \subset \rho(\mathcal{A})$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $R(\lambda,\mathcal{A}) = (\lambda I - \mathcal{A})^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.*

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time), that is to say:

**Definition 8** [7] *An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$, there exists a constant $k(\tau) \geq 0$ such that*

$$\|S(t) - S(s)\| \leq k(\tau)\,|t-s|\,,\ \text{for all } t,s \in [0,\tau]\,.$$

**Theorem 9** [21] *Assertions **(i)** and **(ii)** are equivalent :*
***(i)*** *A is the generator of a locally Lipschitz continuous integrated semigroup,*
***(ii)*** *A satisfies the condition (**HY**).*

## 3 Semigroup associated to a nonautonomous ordinary differential equation

In this section, we recall the construction of an extrapolation space introduced by Da Prato-Grisvard [9] and Nagel [25], which makes it possible to go from a non-autonomous equation to an autonomous one. We will point out the construction and properties of the semigroup $(T(t))_{t\geq 0}$ associated with an evolution family arising from a nonautonomous ordinary differential equation. This notion will be useful in the sequel.

We now consider a nonautonomous Cauchy problem in a Banach space $X$

$$\left\{ \begin{array}{lcl} \dfrac{d}{dt}u(t) & = & A(t)u(t) \\ u(s) & = & x \in X \end{array} \right. \tag{9}$$

for $t \geq s \in \mathbb{R}$. $A(t)$ is assumed to be a bounded linear operator, such that for $t \rightarrow A(t)$ is continuous, from $\mathbb{R}$ into $\mathcal{L}(X)$.
We denote

$$BUC(\mathbb{R}, X) = \{f : \mathbb{R} \rightarrow X : f \text{ is uniformly continuous and bounded.}\},$$

with the norm

$$\|f\| = \sup_{x\in\mathbb{R}} |f(x)|.$$

We consider the operator $\mathcal{A}$ on $UBC(\mathbb{R}, X)$ associated with equation (9), defined by :

$$(\mathcal{A}f)(s) = -f^{'}(s) + A(s)f(s)$$

with domain

$$D(\mathcal{A}) = \left\{ f \in BUC(\mathbb{R}, X),\ f \text{ is differentiable and } f^{'} \in BC(\mathbb{R}, X) \right\}.$$

**Theorem 10** *We suppose that for any $t$, $A(t)$ is a linear bounded operator, such that $t \rightarrow A(t)$ is continuous and uniformly bounded, from $\mathbb{R}$ into $\mathcal{L}(X)$. Then, the operator $\mathcal{A}$ generates a strongly continuous semigroup $T(t)$ in $BUC(\mathbb{R}, X)$.*

**Proof :** In order to determine the resolvent operator, we must solve the equation

$$\left\{ \begin{array}{l} (\lambda I - \mathcal{A})^{-1} f = w \\ w \in D(\mathcal{A}) \end{array} \right.$$

where $f \in BUC(\mathbb{R}, X)$.
Clearly, $w$ depends on $\lambda$. Occasionally, we will use the notation $w_\lambda$.
The following formula can be obtained by standard computations

$$w_\lambda(s) = \int_{-\infty}^{s} U(s,t)f(t)e^{\lambda(t-s)}dt,$$

where $(U(s,t))_{s\geq t}$ is an evolution family satisfying

$$\|U(s,t)\| \leq Me^{\omega(s-t)} \text{ for some } M \geq 1, \text{ and } \lambda \geq \omega \in \mathbb{R}.$$

To show the Hille-Yosida property, it is necessary here to consider the $n^{th}$ iterates of $(\lambda I - \mathcal{A})^{-1}$.
We can show that

$$[(\lambda I - \mathcal{A})^{-n} f](s) = \int_{-\infty}^{s} U(s,\sigma) f(\sigma) e^{\lambda(\sigma - s)} \frac{(s-\sigma)^n}{n!} d\sigma,$$

from which we deduce, for $\lambda > \omega$

$$\left\| (\lambda I - \mathcal{A})^{-n} f \right\| \leq \frac{M}{(\lambda - \omega)^n} \left\| f \right\|.$$

To conclude the proof of theorem 10, it remains to be proven that $D(\mathcal{A})$ is dense in $BUC(\mathbb{R}, X)$. But, $D(\mathcal{A})$ contains obviously $C_b^1(\mathbb{R}, X)$, the space of differentiable functions which are bounded from $\mathbb{R}$ into $X$, (since any function in $BUC(\mathbb{R}, X)$ is transformed into such an element by convolution with a function in $\mathcal{D}(\mathbb{R}, X)$, the space of functions infinitely many times differentiable from $\mathbb{R}$ into $X$ with bounded support), and this set is obviously dense in $BUC(\mathbb{R}, X)$.□

## 4 Ordinary differential equation with impulses

We first consider an ordinary differential equation with impulses

$$\begin{cases} \dfrac{du}{dt} = Au(t),\ t > \sigma,\ t \neq t_i,\ i \in \mathbb{Z},\ \sigma \in \mathbb{R}\ (DE) \\ u(\sigma) = \xi \in X,\ (IC) \\ u(t_i^+) = B_i u(t_i^-),\ u(t_i^-) = u(t_i),\ t_i \geq \sigma,\ i \in \mathbb{Z},\ (IMC) \end{cases} \tag{10}$$

where
$(H_1)$-$A$ is a bounded linear operator,
$(H_2)$-$(B_i)_{i\in\mathbb{Z}}$ is a family of uniformly bounded linear operators, ($\|B_i\| \leq M$, $\forall i$, $M$ is a constant)
$(H_3)$-$(t_i)_{i\in\mathbb{Z}}$ is an increasing family of real numbers, and there exist $\delta > 0$ and $T < \infty$, such that for any $i \in \mathbb{Z}$,

$$0 < \delta \leq t_{i+1} - t_i \leq T < \infty. \tag{11}$$

We first introduce the index function $i(\sigma) = \min\{j : t_j \geq \sigma\}$ where, for each $\sigma$, the impulse condition reads in terms of $i(\sigma)$ as

$$u(t_i^+) = B_i u(t_i),\ i \geq i(\sigma)$$

If $\sigma = t_{i(\sigma)}$, that is to say, if we start from an impulse time point, then we use $u(\sigma)$ for $u(\sigma^-)$.
We denote

$$BR(\mathbb{R}, X) = \left\{ \begin{array}{l} f : \mathbb{R} \to X,\ f \text{ is regulated, continuous to} \\ \text{the left and bounded in } \mathbb{R}. \end{array} \right\}. \tag{12}$$

We consider the family $U(t,s)_{t\geq s}$ associated to (10) defined as follows:

$$U(t,s) = \begin{cases} e^{(t-s)A} & \text{if} \quad [s,t[ \cap D = \emptyset, \\ e^{(t-t_i)A} \circ B_i \circ e^{(t_i - s)A} & \text{if} \quad [s,t[ \cap D = \{t_i\}, \\ Id & \text{if} \quad t = s, \end{cases} \tag{13}$$

**Remark 1**: $U(t,s)$ is not fully defined by (13). Extension of $U(t,s)$ over the whole set $D$ is obtained by using the chain rule property. For arbitrary $t, s,\ t \geq s$ we define $U(t,s)$ as a finite product of operators $U(\tau_{j+1}, \tau_j)$, where $s = \tau_0 < \tau_1 < ... <$

$\tau_j < \tau_{j+1} < ... < \tau_{N+1} = t$ and $U(\tau_{j+1}, \tau_j)$ is given by formula (13), for each $j$, $0 \leq j \leq N$. We give the following definition.

**Definition 11** *Let $X$ be a Banach space. A one parameter family $(T(t))_{t\geq 0}$, of bounded linear operators on $BR(\mathbb{R}, X)$ is a pointwise regulated semigroup if :*
*(i) $(T(t))_{t\geq 0}$ is an algebraic semigroup.*
*(ii) for any fixed $f \in BR(\mathbb{R}, X)$, $s \in \mathbb{R}$ and $t \geq 0$, both the maps :*

$$\text{and} \quad \left.\begin{array}{l} t \to (T(t)f)(s) \\ s \to (T(t)f)(s) \end{array}\right\} \text{ are regulated.}$$

The following lemmas will be needed to prove the first result which states that, the operator $(T(t))_{t\geq 0}$ for ordinary (respectively delay) differential equations with impulses, in subsection 4.1 (respectively subsection 5.1), is a pointwise regulated semigroup of bounded linear operators on $BR(\mathbb{R}, X)$ (respectively $BR(\mathbb{R}, R_X^+)$).

**Lemma 12** *The evolution family $U(t, s)$, for all $t \geq s$ associated to (10), can be represented by the product of all $U(\tau_{j+1}, \tau_j)$, $\tau_j < \tau_{j+1}$, $0 \leq j \leq N$, and we have:*
•*If $[s, t[ \cap D = \emptyset$,*

$$U(t, s) = e^{(t-s)A} \tag{14}$$

•*If $[s, t[ \cap D = \{t_p\}, p \in \mathbb{Z}$*

$$U(t, s) = e^{(t-t_p)A} \circ B_p \circ e^{(t_p-s)A} \tag{15}$$

•*If $[s, t[ \cap D = \{t_i, i = p, p+1, ..., q; q > p, (p, q) \in \mathbb{Z}^2\}$,*

$$U(t, s) = e^{(t-t_q)A} \circ B_q \circ e^{(t_q-t_{q-1})A} \circ ... \circ B_p \circ e^{(t_p-s)A}. \tag{16}$$

**Proof:** We consider the general case when $[s, t[ \cap D$ has $q - p + 1$ elements with $q > p$, then we have

$$[s, t[ \cap D = \{t_i, i = p, p+1, ..., q; q > p, (p, q) \in \mathbb{Z}^2\},$$

and if we consider a finite family $(\tau_l)_{0\leq l\leq N+1}$, such that $\tau_l < \tau_{l+1}$ $s = \tau_0$, $\tau_{N+1} = t$. In order for $U(\tau_{l+1}, \tau_l)$ to be defined by (13), it is necessary that if for some $l$ and $p$ we have: $t_p \in [\tau_l, \tau_{l+1}[$, then, for this $l$, $[\tau_l, \tau_{l+1}[ \cap D = \{t_p\}$. So, in view of (13), we will have:
if $[\tau_l, \tau_{l+1}[ \cap D = \emptyset$

$$U(\tau_{l+1}, \tau_l) = e^{(\tau_{l+1}-\tau_l)A},$$

or, if $[\tau_l, \tau_{l+1}[ \cap D = \{t_p\}$

$$U(\tau_{l+1}, \tau_l) = e^{(\tau_{l+1}-t_p)A} \circ B_p \circ e^{(t_p-\tau_l)A}.$$

Taking the product of $U(\tau_{l+2}, \tau_{l+1})$ and $U(\tau_{l+2}, \tau_{l+1})$, we obtain:
if $[\tau_l, \tau_{l+2}[ \cap D = \{t_p\}$

$$U(\tau_{l+2}, \tau_{l+1}) = e^{(\tau_{l+2}-\tau_{l+1})A},$$

or
if $[\tau_l, \tau_{l+2}[ \cap D = \{t_p, t_{p+1}\}$

$$U(\tau_{l+2}, \tau_l) = e^{(\tau_{l+2}-t_{p+1})A} \circ B_{p+1} \circ e^{(t_{p+1}-t_p)A} \circ B_p \circ e^{(t_p-\tau_l)A}.$$

We can similarly represent the product of all the $(U(\tau_{l+1}, \tau_l))_{0\leq l\leq N}$. We arrive at

$$U(t, s) = e^{(t-t_q)A} \circ B_q \circ e^{(t_q-t_{q-1})A} \circ ... \circ B_p \circ e^{(t_p-s)A}.$$

Obviously, this expression is independent of the family $(\tau_l)_{0\le l\le N+1}$, and we obtained a doubly indexed family of operators $U(t,s)$ which satisfies the chain rule property

$$U(t,s)=U(t,r)\circ U(r,s),$$

for all $t,r,s$ such that $t\ge r\ge s$.□

We note in this section, that the limit is not uniform but, is only a **pointwise limit.**

**Lemma 13** *Consider equation (10) with $(B_i)_{i\in\mathbb{Z}}$ and $D$ satisfying assumptions $(H_2)$ and $(H_3)$, and $U(t,s)$ given as said above (14)-(16). Then, $t\to U(t,s)$ is continuous at any point $t\notin D$, and continuous to the left at any $t_i$, and we have*

$$\lim_{\substack{t\to t_i\\ t>t_i}}U(t,s)=B_i\lim_{\substack{t\to t_i\\ t<t_i}}U(t,s)=B_iU(t_i,s). \tag{17}$$

**Proof:** Let $U(t,s)$ be the evolution family associated to (10), and suppose first that $s,t\in\mathbb{R}$, be such that $0<t-s<\delta$. Then :
•If $s<t$ and $t_i\notin[s,t[$ then : $U(t,s)=\exp((t-s)A)$. Consequently, $t\to U(t,s)$ is continuous.
•If $t_i\in[s,t[$ , in view of (11), it is the only point of the set $\{t_i : i\in Z\}$. Then

$$U(t,s)=\exp((t-t_i)A)\circ B_i\circ\exp((t_i-s)A).$$

Thus,

$$\lim_{\substack{t\to t_i\\ t>t_i}}U(t,s)=B_i\lim_{\substack{t\to t_i\\ t<t_i}}U(t,s)=B_iU(t_i,s).$$

In order to extend the property to an arbitrary pair $(t,s)$, $t>s$, we just have to express $U(t,s)$ a product

$$U(t,s)=U(t,\tau^*)\circ U(\tau^*,s)$$

where we assume that $0<t-\tau^*<\delta$. Then, the first step yields continuity of $t\to U(t,\tau^*)$ at any $t\notin D$, and

$$\lim_{\substack{t\to t_i\\ t>t_i}}U(t,\tau^*)=B_i\lim_{\substack{t\to t_i\\ t<t_i}}U(t,\tau^*)=B_iU(t_i,\tau^*).$$

Taking the product of the above with $U(\tau^*,s)$, formula (17) extends to any pair $(t,s)$, $t>s$.□

**4.1 Pointwise regulated semigroup.** We will now show that the semigroup $(T(t))_{t\ge0}$ defined by (8) associated with the evolution family $U(t,s)$, defined by formula (14)-(16) in lemma 12, is pointwise regulated. We have $(T(t))_{t\ge0}$ defined, for any $f\in BR(\mathbb{R},X)$, $s\in\mathbb{R}$, $t\ge0$, as follows :
•If $[s-t,s[\cap D=\emptyset$

$$(T(t)f)(s)=e^{tA}f(s-t), \tag{18}$$

•If $[s-t,s[\cap D=\{t_n\}\,,n\in\mathbb{Z}$

$$(T(t)f)(s)=e^{(s-t_n)A}\circ B_n\circ e^{(t_n-(s-t))A}f(s-t), \tag{19}$$

•If $[s-t,s[\cap D=\{t_i, i=n,n+1,...,m; m>n,(n,m)\in\mathbb{Z}^2\}$,

$$(T(t)f)(s)=e^{(s-t_m)A}\circ B_m\circ e^{(t_m-t_{m-1})A}\circ...\circ B_n\circ e^{(t_n-(s-t))A}f(s-t). \tag{20}$$

From the last expression (20) and $(H_1)-(H_3)$, we have the following estimation:

$$\begin{aligned} \|T(t)f\| &= \sup_s |(T(t)f)(s)| \\ &\leq e^{t\|A\|}\sup_s(\prod_{s-t<t_i<s}\|B_i\|)\sup_s(\sup_{s-t<\tau<s}|f(\tau)|) \\ \|T(t)f\| &\leq M_t e^{t\|A\|}\,\|f\| \end{aligned} \tag{21}$$

where $M_t=\sup_s(\prod_{s-t<t_i<s}\|B_i\|)$, $\|f\|=\sup_s(\sup_{s-t<\tau<s}|f(\tau)|)$.
Let us now prove our first result:

**Theorem 14** *$(T(t))_{t\geq 0}$ is a pointwise regulated semigroup of bounded linear operators on $BR(\mathbb{R},X)$.*

**Proof**: We first point out that -due to semigroup properties- $(T(t))_{t\geq 0}$ being pointwise regulated is equivalent to $(T(t)f)(s)$ being separately regulated in $t$ and $s$, for $0\leq t<\tau$, and all $s\in\mathbb{R}$, for some $\tau>0$.
So, we will assume for the rest of the proof that $0\leq t<\delta$ (where $\delta$ is as in ($H_3$)).
For $f\in BR(\mathbb{R},X)$, we have

$$(T(t)f)(s)=U(s,s-t)[f(s-t)],\ \forall t\geq 0.$$

•We fix $t\geq 0$, then we have for : $t_i-\delta<s\leq t_i$,

$$(T(t)f)(s)=e^{tA}f(s-t).$$

In view of $(H_1)$, then $s\to(T(t)f)(s)$ is regulated.
For : $s-t\leq t_i<s$, we have

$$(T(t)f)(s)=e^{(s-t_i)A}\circ B_i\circ e^{(t_i-(s-t))A}f(s-t).$$

From this expression we can using $(H_1)$ and $(H_2)$ we have that $s\to(T(t)f)(s)$ is regulated at every $s\neq t_i$. We now consider the situation at $s=t_i$ then

$$\lim_{\substack{s\to t_i\\ \leq}}(T(t)f)(s)=e^{tA}\lim_{\substack{s\to t_i\\ \leq}}f(s-t)$$

and

$$\lim_{\substack{s\to t_i\\ >}}(T(t)f)(s)=B_ie^{tA}\lim_{\substack{s\to t_i\\ >}}f(s-t).$$

Therefore, $s\to(T(t)f)(s)$ is regulated over $\mathbb{R}$.
•In the same way, for fixed $s$, we have

$$(T(t)f)(s)=\begin{cases} e^{At}f(s-t), & \text{if}\quad [s-t,s[\cap D=\emptyset, \\ e^{(s-t_i)A}\circ B_i\circ e^{(t_i-(s-t))A}f(s-t) & \text{if}\quad [s-t,s[\cap D=\{t_i\}. \end{cases}$$

Then, $t\to(T(t)f)(s)$ is regulated, for all $0\leq t<\delta$.
By using proprietie of algebraic semigroup, for fixed $t\in[0,\delta]$, and for small $h$, we have

$$(T(t+h)f)(s)=(T(t)T(h)f)(s),\ \text{for } f\in BR(\mathbb{R},X),$$

since, from the first step we have that, for small $h$, $T(h)$ is a pointwise regulated semigroup of bounded linear operators on $BR(\mathbb{R},X)$.
Then, we have the result for all $t\geq 0$.

**Remark 2**: The map $(t,s)\to(T(t)f)(s)$ is continuous at any point $(t,s)$, $t\geq 0$, $s\in\mathbb{R}$, such that neither $s$ nor $s-t$ is in the set $D_f\cup\{t_i\}$, where $D_f$ is the set of the points of discontinuity of $f$. Then, $(t,s)\to(T(t)f)(s)$ is continuous outside of a negligible set in $\mathbb{R}^+\times\mathbb{R}$.

**4.2 Integrated semigroup.** We showed in Theorem 14 that the map $t \to (T(t)f)(s)$ is regulated for every $s$. As a consequence, we can integrate the function in $t$ for, each fixed $s$. This yields an operator $S(t)$ on $BR(\mathbb{R}, X)$ given by

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau,$$

for $t \geq 0$, $f \in BR(\mathbb{R}, X)$, and $T(t)$ given by (18)-(20) for all $s \in \mathbb{R}$.
$S(t)f$ is well defined by formula (7). We note that this expression is not defined as a vector integral on $BR(\mathbb{R}, X)$ but as a function defined at each point $s$ as the integral of a regulated function.
We now prove the propriety of integrated semigroup:

**Theorem 15** $(S(t))_{t>0}$ *defined by formula (7) is a norm continuous integrated semigroup on* $BR(\mathbb{R}, X)$.

**Proof**: Let $f \in BR(\mathbb{R}, X)$, then for small $h$, we have for any $t \in \mathbb{R}^+$

$$S(t+h)f - S(t)f = \int_t^{t+h} T(\tau)fd\tau. \tag{22}$$

From (21), we have,

$$\|T(\tau)f\| \leq M_{t+1}e^{(t+1)\|A\|}\|f\|, \text{ for any } \tau \in [t, t+h],$$

where $M_t = \sup_s(\prod_{s-t<t_i<s} \|B_i\|)$.
Thus,

$$\|S(t+h)f - S(t)f\| \leq M_{t+1}e^{(t+1)\|A\|}h\|f\|,$$

which shows the norm continuity.
For the algebraic formula, we have for any $f \in BR(\mathbb{R}, X)$, $\eta \in \mathbb{R}$,

$$\begin{aligned}
\int_0^s (S(\tau+\sigma) - S(\sigma))(f)(\eta)d\sigma &= \int_0^s ((S(t+\sigma) - S(\sigma))f)(\eta)d\sigma \\
&= \int_0^s (\int_0^{t+\sigma} (T(\tau)f)(\eta)d\tau - \\
&\quad \int_0^{\sigma} (T(\tau)f)(\eta)d\tau)d\sigma \\
&= \int_0^s (\int_\sigma^0 (T(\tau)f)(\eta)d\tau + \\
&\quad \int_0^{t+\sigma} (T(\tau)f)(\eta)d\tau)d\sigma \\
&= \int_0^s \int_\sigma^{t+\sigma} (T(\tau)f)(\eta)d\tau d\sigma \\
&= \int_0^s \int_0^t (T(\tau+\sigma)f)(\eta)d\tau d\sigma \\
&= \int_0^s \int_0^t (T(\tau)T(\sigma)f)(\eta)d\tau d\sigma \\
&= \int_0^s (T(\tau)\int_0^t T(\sigma)f)(\eta)d\sigma d\tau
\end{aligned}$$

$$\begin{aligned} &= \int_0^s (T(\tau)S(t)f)(\eta)d\tau \\ &= \int_0^s T(\tau)S(t)(f)(\eta)d\tau \\ &= (S(s)S(t)f)(\eta). \end{aligned}$$

Then

$$S(s)S(t) = \int_0^s (S(t+\sigma) - S(\sigma))d\sigma,$$

and

$$S(0) = 0.$$

Consequently, $S(t)$ is a norm continuous integrated semigroup on $BR(\mathbb{R}, X)$.□

**Theorem 16** *Let $S$ be given by (7), and $f \in BR(\mathbb{R}, X)$. Then, $s \to (S(t)f)(s)$ is continuous at each $s \notin \{t_i\}$ and all $t > 0$ fixed, and we have*

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s) = B_i \lim_{\substack{s \to t_i \\ <}} (S(t)f)(s).$$

**Proof :** Let $f \in BR(\mathbb{R}, X)$. For $t$ fixed such that $0 \leq t < \delta$,
we have

$$(S(t)f)(s) = \int_0^t (T(\tau)f)(s)d\tau.$$

There are 2 cases to consider :

**a)** If $t_i < s < t_i + \delta$ and $s - t \leq t_i$, we have

$$\begin{aligned} (S(t)f)(s) &= \int_0^{s-t_i} (T(\tau)f)(s)d\tau + \int_{s-t_i}^t (T(\tau)f)(s)d\tau \\ &= \int_0^{s-t_i} e^{\tau A} f(s-\tau)d\tau + \\ &\quad \int_{s-t_i}^t e^{(s-t_i)A} \circ B_i \circ e^{(t_i-(s-\tau))A} f(s-t)d\tau. \end{aligned}$$

**b)** If $s < t_{i+1}$ and $s - t > t_i$, then

$$\begin{aligned} (S(t)f)(s) &= \int_0^t (T(\tau)f)(s)d\tau \\ &= \int_0^t e^{\tau A} f(s-\tau)d\tau. \end{aligned}$$

We notice that :

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s) = \int_0^t B_i \exp(\tau A) \lim_{\substack{s \to t_i \\ >}} f(s-\tau)d\tau$$

and

$$\lim_{\substack{s \to t_i \\ <}} (S(t)f)(s) = \int_0^t \exp(\tau A) \lim_{\substack{s \to t_i \\ <}} f(s-\tau)d\tau.$$

Then, $s \to (S(t)f)(s)$ is continuous at each point $s \notin \{t_i\}_{i\in\mathbb{Z}}$ and thus has possible discontinuities only at points $t_k$. By (22) in Theorem 15, we have

$$\begin{aligned} (S(t+h)f)(s) &= (S(t)f)(s) + \int_0^h (T(\tau+t)f)(s)d\tau \\ &= (S(t)f)(s) + \int_0^h (T(\tau)T(t)f)(s)d\tau \\ &= (S(t)f)(s) + (S(h)T(t)f)(s). \end{aligned}$$

By using Theorem 14, $t \to T(t)f$ is regulated, for any $f \in BR(\mathbb{R}, X)$.
Then, we have the result for all $t \geq 0$.□

We remark that the integration of the semigroup made it possible to eliminate all discontinuities due to the data, except at points $s = t_i$, $i \in \mathbb{Z}$.

## 5 Delay differential equations with impulses

Similar to section 4, we suppose that the limit is only the **pointwise limit**. The equation without impulses is given by

$$\frac{du}{dt} = Lu_t,$$

where $L$ is defined on $C_X$ :

$$L : C_X \to X.$$

We suppose that $L$ is linear and continuous on $C_X$, and has a continuous extension to a vector Radon measure on $X$ (see, [11] and [14]) which can be extended to $R_X$. We still denote $L$ the extension to $R_X$.
For example, if for every $x \in X$, the map $f \in C([-r, 0], X) \to L(f \otimes x)$ is weakly compact (see [11] page 182 and [13]), then $L$ can be extended to $R_X$. As a special case, we may consider $L$ defined by

$$L = \sum_{i=1}^{N} l_i \otimes x_i$$

where $l_i \in (C([-r, 0], X))^*$, $x_i \in X$. In fact,

$$L(F) = \sum_{i=1}^{N} (l_i \otimes x_i)(F) = \sum_{i=1}^{N} l_i(F)x_i.$$

For $f \in C([-r, 0], \mathbb{R})$, $x \in X$, we have

$$L(f \otimes x) = \sum_{i=1}^{N} l_i(f \otimes x)x_i$$

then

$$f \to \sum_{i=1}^{N} l_i(f \otimes x)x_i$$

is compact since it is continuous with values in a finite dimensional subspace of $X$, so it is also weakly compact.

The equation in $R_X^+$ is given as follows :

$$\begin{cases} \dfrac{du}{dt} & = & Lu_t \\ u_0 & = & \phi \in R_X \\ u(0^+) & = & x \in X. \end{cases} \tag{23}$$

Problem (23) has, for all $(\phi, x)$, one and only one solution which is a regulated function on $[-r, +\infty[$.
We call $T_L$ the semigroup associated with the equation

$$\frac{du}{dt} = Lu_t \text{ on } R_X^+.$$

Thus, one has, for all $(\phi, x) \in R_X^+$,

$$u_t = T_L(t)(\phi, x). \tag{24}$$

The semigroup $T_L$ is exponentially bounded, so that there exists $\omega_0 \in \mathbb{R}$, such that for each $\omega > \omega_0$, there exists $C \geq 1$ for which

$$\|T_L(t)(\phi, x)\| \leq Ce^{t\omega} \|(\phi, x)\|. \tag{25}$$

**Proposition 17** *Let $(\phi, x) \in R_X^+$. Then, the semigroup $T_L(t)$ associated to (23), defined by formula (24), satisfies the following relationship :*

$$T_L(t)(\phi, x) = \phi_t^0 + H_t^0 \otimes (x - \phi(0)) + (\int_0^{\max(0,\bullet)} L(T_L(s)(\phi, x))ds)_t \tag{26}$$

*where*

$$\phi^0(\theta) = \begin{cases} \phi(\theta) & \theta \leq 0 \\ \phi(0) & \theta > 0 \end{cases} \tag{27}$$

*and $H^0$ is the Heaviside function*

$$H^0(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases} \tag{28}$$

**Proof:** From equation (23), we can write the solution $u$ in the form

$$u(t) = x + \int_0^t L(u_\tau)d\tau.$$

From (24), (27) and (28), we have

$$u_t = T_L(t)(\phi, x) = \phi_t^0 + H_t^0 \otimes (x - \phi(0)) + (\int_0^{\max(0,\bullet)} L(T_L(\tau)(\phi, x))d\tau)_t.\square$$

**Remark 3**: We have $u_t \in R_X$, $\forall t > 0$, and $t \to u_t$ is a regulated function because functions $t \to \phi_t^0$ and $t \to H_t^0$ are regulated.
We now consider the delay differential equation with impulses (1)

$$\begin{cases} \dfrac{du}{dt} = Lu_t,\ t > \sigma,\ t \neq t_i,\ i \in \mathbb{Z},\ \sigma \in \mathbb{R} \\ u_0(t) = \phi(t - \sigma),\ \sigma - r \leq t \leq \sigma \\ u(t_i^+) = B_i u(t_i^-),\ u(t_i^-) = u(t_i),\ t_i \geq \sigma,\ i \in \mathbb{Z}, \end{cases}$$

where
a)-$(B_i)_{i\in\mathbb{Z}}$ and $(t_i)_{i\in\mathbb{Z}}$ satisfy $(H_2)$ and $(H_3)$.
b)-$\phi \in R_X$.

We introduce the evolution family $U(t,s)$ associated with (1) in the following way :

$$U(t,s)=\begin{cases} T_L(t-s) & \text{if} \quad [s,t[\cap D=\emptyset, \\ T_L(t-t_i)\circ \widetilde{B}_i\circ T_L(t_i-s) & \text{if} \quad [s,t[\cap D=\{t_i\}, \\ Id & \text{if} \quad t=s, \end{cases} \tag{29}$$

where, for each $B_i \in \mathcal{L}(X)$, we denote

$$\begin{array}{rlll} \widetilde{B}_i: & R_X^+ & \rightarrow & R_X^+ \\ & (\phi,x) & \rightarrow & (\phi,B_i x), \end{array} \tag{30}$$

more exactly from (2), we have

$$\widetilde{B}_i(T_L(t_i-s)(\phi,x))=(T_L(t_i-s)(\phi,x),B_i\left[(T_L(t_i-s)(\phi,x))(0)\right]).$$

**Remark 4:** using the same technique as explained in **Remark 1** for the ordinary case, we can extend $U(t,s)$ to all pairs $(t,s)$ , by using a finite product of operators $U(\tau_{j+1},\tau_j)$, where $s=\tau_0<\tau_1<...<\tau_j<\tau_{j+1}<...<\tau_{N+1}=t$ and $U(\tau_{j+1},\tau_j)$ is given by formula (29), for each $j$, $0\leq j\leq N$, and we have the following lemma.

**Lemma 18** *The evolution family $U(t,s)$, for all $t\geq s$ associated to (1), can be represented by the product of all $U(\tau_{j+1},\tau_j)$, $\tau_j<\tau_{j+1}$, $0\leq j\leq N$, and we have:*
•*If* $[s,t[\cap D=\emptyset$,

$$U(t,s)=T_L(t-s) \tag{31}$$

•*If* $[s,t[\cap D=\{t_p\}\,,p\in\mathbb{Z}$

$$U(t,s)=T_L(t-t_p)\circ\widetilde{B}_p\circ T_L(t_p-s) \tag{32}$$

•*If* $[s,t[\cap D=\{t_i,i=p,p+1,...,q;q>p,(p,q)\in\mathbb{Z}^2\}$,

$$U(t,s)=T_L(t-t_q)\circ\widetilde{B}_q\circ T_L(t_q-t_{q-1})\circ...\circ\widetilde{B}_p\circ T_L(t_p-s). \tag{33}$$

**Proof:** We use the same technique as in lemma 12.□

**Remark 5**: Hypothesis $(H_2)$, Proposition 17 and **Remark 3** together yield that, for an arbitrary $t>s$, $U(t,s)(\phi,x)\in R_X$.

**5.1 Pointwise regulated semigroup.** Let $f\in BR(\mathbb{R},R_X^+)$. So, $f=(\phi,x)$, where $\phi:\mathbb{R}\rightarrow R_X$, $x:\mathbb{R}\rightarrow X$, and we have: $f(s)=(\phi(s),x(s))$, for all $s\in\mathbb{R}$, where the functions $s\rightarrow\phi(s)$ and $s\rightarrow x(s)$ are bounded and regulated, continuous to the left. Moreover, for every $s\in\mathbb{R}$, the function $\theta\rightarrow\phi(s)(\theta)$ is also regulated continuous on the left, and we denote by $\phi^+(\theta^{'})=\lim\limits_{\substack{\theta\rightarrow\theta^{'}\\ >}}\phi(\theta)$ the **pointwise limit** to the right at $\theta$, and $\phi(0)=\lim\limits_{\substack{\theta\rightarrow 0\\ \leq}}\phi(\theta)$ where $\phi\in R_X$, is a regulated function continuous to the left.
For each $B_i\in\mathcal{L}(X)$, we denote by

$$\begin{array}{rlll} \widetilde{B}_i^+: & R_X^+ & \rightarrow & R_X^+ \\ & (\phi,x) & \rightarrow & (\phi^+,B_i x). \end{array} \tag{34}$$

**Note**: In this section, the limit is not uniform but only a **pointwise limit**.

**Lemma 19** *Consider equation (1) with $(B_i)_{i\in\mathbb{Z}}$ and $D=\{t_i,i\in\mathbb{Z}\}$ satisfying the assumptions $(H_2)$ and $(H_3)$, and $U(t,s)$ given by (31)-(33). Then, for each*

*$(\phi, x) \in R_X^+$, $t \to U(t,s)(\phi,x)$ is a regulated function on $\mathbb{R}^+$, and we have at, $t_i \in D$:*

$$\lim_{\substack{t \to t_i \\ t > t_i}} (U(t,s)(\phi,x)) = \widetilde{B}_i^+ (U(t_i,s)(\phi,x)) \tag{35}$$

*where for each $B_i \in \mathcal{L}(X)$, $\widetilde{B}_i^+$ is defined by (34).*

**Proof :** Let $U(t,s)$ be the evolution family defined by formula (29). For all $s, t$, $0 < t - s < \delta$, we have :
a) If $s < t$ and $t_i \notin [s,t[$, then from Proposition 17 and **Remark 3** : $t \to U(t,s)(\phi,x) = T_L(t-s)(\phi,x)$, is a regulated function on $\mathbb{R}^+$.
b) If $t_i \in [s,t[$, for an integer $i$, as $\inf_{j \in \mathbb{Z}} (t_{j+1} - t_j) = \delta > 0$, then $[s,t[$ contains only one point of $\{t_k : k \in Z\}$, then

$$t \to U(t,s)(\phi,x) = T_L(t-t_i) \circ \widetilde{B}_i \circ T_L(t_i - s)(\phi,x).$$

where $\widetilde{B}_i$ is defined by (30).
Then

$$\begin{aligned} \lim_{\substack{t \to t_i \\ t > t_i}} (U(t,s)(\phi,x)) &= \lim_{\substack{t \to t_i \\ t > t_i}} \Big( T_L(t-t_i) \circ \widetilde{B}_i \circ T_L(t_i - s)(\phi,x) \Big) \\ &= \Big( (T_L(t_i - s)(\phi,x))^+, B_i[(T_L(t_i - s)(\phi,x))(0)] \Big) \\ &= \Big( (U(t_i,s)(\phi,x))^+, B_i[(U(t_i,s)(\phi,x))(0)] \Big) \\ &= \widetilde{B}_i^+ (U(t_i,s)(\phi,x)). \end{aligned}$$

In order to extend the property to an arbitrary pair $(t,s)$, $t > s$, we just have to express $U(t,s)$ as product

$$U(t,s)(\phi,x) = U(t,\tau^*) \circ U(\tau^*,s)(\phi,x)$$

where we assume that $0 < t - \tau^* < \delta$. Then, from **Remark 5,** we have

$$U(\tau^*,s)(\phi,x) \in R_X,$$

and, also the first step yields that $t \to U(t,s)(\phi,x)$ is a regulated function on $\mathbb{R}^+$, and

$$\lim_{\substack{t \to t_i \\ t > t_i}} U(t,\tau^*) = B_i \lim_{\substack{t \to t_i \\ t < t_i}} U(t,\tau^*) = B_i U(t_i,\tau^*).$$

Taking the product of the above with $U(\tau^*,s)$, formula (35) extends to any pair $(t,s)$, $t > s$.$\square$

We will now show that the semigroup $(T(t))_{t \geq 0}$ defined by (8) associated with the evolution family $U(t,s)$, defined by formula (31)-(33) in lemma 18, is pointwise regulated. We have $(T(t))_{t \geq 0}$ defined, for any $f \in BR(\mathbb{R}, R_X^+)$, $s \in \mathbb{R}$, $t \geq 0$, as follows :
•If $[s-t,s[ \cap D = \emptyset$,

$$T(t)f)(s) = T_L(t)[f(s-t)]$$

•If $[s-t,s[ \cap D = \{t_n\}, n \in \mathbb{Z}$

$$(T(t)f)(s) = T_L(s-t_n) \circ \widetilde{B}_n \circ T_L(t_n - (s-t))[f(s-t)]$$

•If $[s-t,s[ \cap D = \{t_i, i = n, n+1, ..., m; m > n, (n,m) \in \mathbb{Z}^2\}$,

$$(T(t)f)(s) = T_L(s-t_m) \circ \widetilde{B}_m \circ T_L(t_m - t_{m-1}) \circ ... \circ \widetilde{B}_n \circ T_L(t_n - (s-t))[f(s-t)].$$

From the last expression (6), (25) and $(H_1)-(H_3)$, we have the following estimation:

$$\begin{aligned}\|T(t)f\| &= \sup_s |(T(t)f)(s)| \\ &\leq e^{t\|A\|}\sup_s(\prod_{s-t<t_i<s}\left\|\widetilde{B}_i\right\|)\sup_s(\sup_{s-t<t_p<s}|f(\tau)|) \\ \|T(t)f\| &\leq CM_te^{t\omega}\|f\| \qquad (36)\end{aligned}$$

where $M_t=\sup_s(\prod_{s-t<t_i<s}\left\|\widetilde{B}_i\right\|)$, $\|f\|=\sup_s(\sup_{s-t<\tau<s}|f(\tau)|)$.

**Theorem 20** *Let $f\in BR(\mathbb{R},R_X^+)$, then $s\to(T(t)f)(s)$ is regulated on $\mathbb{R}$, for all $t\geq 0$.*

**Proof**: Let $f\in BR(\mathbb{R},R_X^+)$, and $0\leq t<\delta$. Using formulas (4)-(5), we see that

$$\lim_{\substack{s\to t_i\\ \leq}}(T(t)f)(s)=T_L(t)\lim_{\substack{s\to t_i\\ \leq}}f(s-t)$$

and

$$\lim_{\substack{s\to t_i\\ >}}(T(t)f)(s)=\widetilde{B}_iT_L(t)\lim_{\substack{s\to t_i\\ >}}f(s-t).$$

As $f$ is regulated, both limits exist. Thus, $s\to(T(t)f)(s)$ is regulated on $\mathbb{R}$, for $0\leq t<\delta$. By using the same technique as in Theorem 14, we have the result for all $t\geq\delta>0$.□

**Theorem 21** *Let $f\in BR(\mathbb{R},R_X^+)$. Then, for $s$ fixed, and $\forall\theta\in[-r,0]$, the map $t\to(T(t)f)(s)(\theta)$ is regulated on $\mathbb{R}^+$. $(f=(\phi,x))$*

**Proof:** Let $f\in BR(\mathbb{R},R_X^+)$. We observe that, for $s$ fixed, $0\leq t<\delta$, if:
a) $t_i-\delta<s\leq t_i$ for a certain $i$, then from (4) we have

$$t\to(T(t)f)(s)(\theta)=(T_L(t)f(s-t))(\theta).$$

Consequently, using formula (26), we have

$$\begin{aligned}(T(t)f)(s)(\theta) &= \phi^0(t+\theta)(s-t)+H^0(t+\theta)\otimes(x(s-t)-\phi(s-t)(0)) \\ &\quad +(\int_0^{\max(0,t+\theta)}L(T_L(\tau)(\phi(s-\tau),x(s-\tau)))d\tau).\end{aligned}$$

Each function $t\to\phi^0(t+\theta)(s-t)$, $t\to H^0(t+\theta)$, $t\to x(s-t)$, $t\to\phi(s-t)$ of the second member of the last equality is regulated on $\mathbb{R}^+$, then $t\to(T(t)f)(s)(\theta)$ is also regulated on $\mathbb{R}^+$.
b) $t_i<s\leq t_{i+1}-\delta$, for some $i$. Then, we have two cases to discuss. If:
•$s-t\leq t_i$, the map $(T(t)f)(s)(\theta)$, is reduced to

$$t\to(T(t)f)(s)(\theta)=\left(T_L(s-t_i)\circ\widetilde{B}_i\circ T_L(t_i-(s-t))f(s-t)\right)(\theta). \qquad (37)$$

Thus, from (26), and if we denote by

$$t\to(\Theta_if)(t)=T_L(t_i-t)f(t), \qquad (38)$$

we have

$$t \to (T(t)f)(s)(\theta) = \begin{cases} \left(\widetilde{B}_i \circ (\Theta_i f)(s-t)\right)^0_{s-t_i}(\theta)+ \\ H^0_{s-t_i}(\theta) \otimes \{B_i\left((\Theta_i(\phi,x))(s-t)(0)\right) - \\ -(\Theta_i(\phi,x))(s-t)(0)\}+ \\ \left(\int_0^{\max(0,\theta)} L(T_L(\tau)\circ \widetilde{B}_i \circ (\Theta_i f)(s-t))d\tau\right)_{s-t_i}. \end{cases} \tag{39}$$

Since, from (26) and (38), we can write

$$\begin{aligned} (\Theta_i f)(s-t) &= T_L(t_i-(s-t))(\phi(s-t),x(s-t)) \\ &= \phi^0_{t_i-(s-t)} + H^0_{t_i-(s-t)} \otimes (x(s-t)-\phi(s-t)(0)) \\ &\quad +(\int_0^{\max(0,\bullet)} L(T_L(\tau)(\phi,x))d\tau)_{t_i-(s-t)} \end{aligned}$$

then $t \to (\Theta_i f)(s-t)$ is regulated on $\mathbb{R}^+$, and we have $B_i$ is bounded then $t \to B_i\left((\Theta_i(\phi,x))(s-t)(0)\right)$ and $t \to (\Theta_i(\phi,x))(s-t)(0)$ are regulated on $\mathbb{R}^+$.
Therefore each function of the second member of equality (39) is regulated on $\mathbb{R}^+$, then in this case also $t \to (T(t)f)(s)(\theta)$ is regulated on $\mathbb{R}^+$.

$\bullet s-t > t_i$, the map $t \to (T(t)f)(s)(\theta)$, is reduced to

$$\begin{aligned} (T(t)f)(s)(\theta) &= (T_L(t)f(s-t))(\theta) \\ &= \phi^0(t+\theta)(s-t) + H^0(t+\theta) \otimes (x(s-t)-\phi(s-t)(0)) \\ &\quad +(\int_0^{\max(0,t+\theta)} L(T_L(\tau)(\phi(s-\tau),x(s-\tau)))d\tau). \end{aligned} \tag{40}$$

Just as in the first case, we have that the map $t \to (T(t)f)(s)(\theta)$ is regulated on $0 \le t < \delta$.
By using the same technique as in the proof of Theorem 14, we have the result for all $t \ge \delta > 0$.□

**5.2 Integrated semigroup.** We consider the family of operators $S(t)$ defined by :

$$(S(t)f)(s)(\theta) = \int_0^t (T(\tau)f)(s)(\theta)d\tau, \ t \ge 0, \tag{41}$$

$f \in BR(\mathbb{R}, R_X^+)$, $T(t)$ being defined by (4)-(6). We note that the expression (41), is not defined as a vector integral on $BR(\mathbb{R}, R_X^+)$ but as a function defined at each point $s$ as the integral of a regulated function.

In parallel, we consider the family $S_L(t)$ of the operators defined on $BR(\mathbb{R}, R_X^+)$ arising from the delay differential equation without impulses :

$$(S_L(t)f)(s)(\theta) = \int_0^t (T_L(\tau)f(s-\tau))(\theta)d\tau.$$

The main ingredient of the proof of Theorem 1 -which justifies the use of the integrated semigroup theory- is the following theorem which states that $S_L(t)(\phi,x)(s) \in C_X$, and $S_L(t)(\phi,x)(s)$ is strongly continuous on $BR(\mathbb{R}, R_X^+)$, for all $t \ge 0$, $s \in \mathbb{R}$.

**Theorem 22** *For all $(\phi, x) \in BR(\mathbb{R}, R_X^+)$, $(S_L(t)(\phi, x))(s)(\theta)$ takes one of the following two expressions :*

$$(S_L(t)(\phi,x))(s)(\theta) = \begin{cases} \int_0^t \phi(s-\tau)(\tau+\theta)d\tau \\ + \int_0^t (\int_0^{\max(0,\bullet)} L(u_\sigma(s-\tau)d\sigma)_\tau d\tau, \ \textit{for } \theta \le -t, \\ \\ \int_0^{-\theta} \phi(s-\tau)(\tau+\theta)d\tau + \int_{-\theta}^t \phi(s-\tau)(0)d\tau + \\ \int_{s-t}^{s+\theta} [x(\sigma) - \phi(\sigma)(0)] \, d\sigma + \\ \int_0^t (\int_0^{\max(0,\bullet)} L(u_\sigma(s-\tau)d\sigma)_\tau d\tau, \ \textit{for } \theta > -t. \end{cases}$$

*(Here $u_t = T_L(t)(\phi, x)$).*
*Moreover, $S_L(t)(\phi, x)(s) \in C_X$, and $S_L(t)(\phi, x)(s)$ is strongly continuous on $BR(\mathbb{R}, R_X^+)$, for all $t \ge 0$, $s \in \mathbb{R}$.*

**Proof** : Let $f = (\phi, x) \in BR(\mathbb{R}, R_X^+)$. Then, we have by definition

$$\begin{aligned} (S_L(t)(\phi,x))(s)(\theta) &= \int_0^t (T_L(\tau)[(\phi,x)(s-\tau)])(\theta)d\tau \\ &= \int_0^t (T_L(\tau)(\phi(s-\tau), x(s-\tau)))(\theta)d\tau. \end{aligned}$$

Consequently, in view of (26), we have

$$(S_L(t)(\phi,x))(s)(\theta) = \begin{cases} \int_0^t \phi_\tau^0(s-\tau)(\theta)d\tau \\ + \int_0^t (H_\tau^0 \otimes (x(s-\tau) - \phi(s-\tau)(0)))(\theta)d\tau \\ + \int_0^t \left( \int_0^{\max(0,\theta)} L(u_\sigma(s-\tau)d\sigma \right)_\tau d\tau, \end{cases} \tag{42}$$

where $u_t = T_L(t)(\phi, x)$.
We observe that the last expression of the second member of the relation (42) is a continuous function of $\theta$.
To establish the continuity of the map

$$\theta \to (S_L(t)(\phi,x))(s)(\theta),$$

for $t$ and $s$ fixed, it remains to study each of the maps

$$\theta \to \int_0^t (\phi_\tau^0(s-\tau)(\theta)d\tau), \text{ and } \theta \to \int_0^t (H_\tau^0 \otimes (x(s-\tau) - \phi(s-\tau)(0))(\theta)d\tau). \tag{43}$$

As

$$\int_0^t \phi_\tau^0(s-\tau)(\theta)d\tau = \int_0^t \phi^0(s-\tau)(\tau+\theta)d\tau,$$

we have

$$\int_0^t \phi_\tau^0(s-\tau)(\theta)d\tau = \begin{cases} \int_0^t \phi(s-\tau)(\tau+\theta)d\tau, & \theta \le -t \\ \int_0^{-\theta} \phi(s-\tau)(\tau+\theta)d\tau + \\ \int_{-\theta}^t \phi(s-\tau)(0)d\tau, & -t < \theta \le 0. \end{cases} \tag{44}$$

We consider the restriction $\phi$ in $[-\xi, \xi] \times [-r, 0]$, $\forall \xi > 0$. Therefore since $\phi \in BR(\mathbb{R}, R_X)$, $\phi$ can be expressed as the **uniform limit of a piecewise constant (in $s$) function**

$$\phi(s)(\theta) = \lim_{\max_{-N<j<N} |J_j| \searrow 0} \text{unif} \sum_j \mathbf{1}_{J_j}(s)\Phi_j(\theta),$$

where $\Phi_j \in R_X$, $J_j$ is an interval of the subdivision, and the length $|J_j|$ of the interval $J_j$, $\mathbf{1}_I$ denotes the indicator function for the set $J$, and similar for each $\Phi_j$ can be expressed as a limit of piecewise constant functions of the form $\sum\limits_k \mathbf{1}_{I_k}(s)\zeta_{jk}$.
Then,

$$\phi(s)(\theta) = \lim_{\max\limits_{-N<j<N}|J_j|\searrow 0,\ \max\limits_{-r<k<0}|I_k|\searrow 0} \text{unif} \sum_{j,k} \mathbf{1}_{J_j}(s) \otimes \mathbf{1}_{I_k}(\theta)\zeta_{jk}, \tag{45}$$

where $\zeta_{jk} \in X$.
We show the continuity of the function defined by (44). In order to do so, we first consider the case when $\phi$ is piecewise constant. Then, we will conclude the property for all $\phi$, using $\mathbf{1}_I \otimes \mathbf{1}_I\zeta$.
Thus,

$$\int_0^t \mathbf{1}_J(s-\tau) \otimes \mathbf{1}_I(\tau+\theta)\zeta d\tau = \int_{s-t}^s \mathbf{1}_I(\tau) \otimes \mathbf{1}_I(s+\theta-\tau)\zeta d\tau. \tag{46}$$

Let us evaluate the difference of the values of (46) at two values $\theta_1$ and $\theta_2$ of $\theta$. Denoting by :

$$E = \int_{s-t}^s \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s+\theta_1-\tau)\zeta d\tau - \int_{s-t}^s \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s+\theta_2-\tau)\zeta d\tau$$

we have

$$\begin{aligned} \|E\| &\leq \int_{s-t}^s \|\mathbf{1}_J(\tau)\| \, \|\mathbf{1}_I(s+\theta_1-\tau) - \mathbf{1}_I(s+\theta_2-\tau)\| \, \|\zeta\| \, d\tau \qquad (47) \\ &\leq \int_{[s-t,s]\cap J} \|\mathbf{1}_I(s+\theta_1-\tau) - \mathbf{1}_I(s+\theta_2-\tau)\| \, \|\zeta\| \, d\tau \\ &\leq 4\,|\theta_1-\theta_2| \, \|\zeta\| \, . \end{aligned}$$

Consequently, the integral function $\int_0^t \phi(s-\tau)(\tau+\theta)d\tau$ is continuous on $[-r,0]$.
Also, in the same way, for $\int_0^{-\theta} \phi(s-\tau)(\tau+\theta)d\tau$, we can make the finite combination of set functions of the type $\mathbf{1}_J(s) \otimes \mathbf{1}_I(\theta)$, then

$$\int_0^{-\theta} \phi(s-\tau)(\tau+\theta)d\tau,$$

is a continuous function on $[-r,0]$.
Obviously, the function $\int_{-\theta}^t \phi(s-\tau)(0)d\tau$ is also continuous in $\theta$, since $\theta$ appears only at one limit of the integral. Moreover, the two formulas of (44) coincide at $\theta = -t$.
Thus the function (44) is continuous in $\theta$. We can write (43) in a different form :

$$\int_0^t (H_\tau^0 \otimes (x(s-\tau) - \phi(s-\tau)(0)))(\theta)d\tau = \begin{cases} 0, & \text{if } \theta \leq -t \\ \int_{s-t}^{s+\theta} [x(\sigma) - \phi(\sigma)(0)]d\tau, & \text{if } \theta > -t. \end{cases} \tag{48}$$

In formula (48), we can easily remark that $\theta$ appears only at one limit of the integral. The two expressions of (48) coincide at $\theta = -t$.
We conclude that $S_L(t)(\phi,x)(s) \in C_X$.
It remains to check that $s \to S_L(t)(\phi,x)(s)$ is continuous. For that we have to look at the first expression $\int_0^t \phi_\tau^0(s-\tau)(\theta)d\tau$, of (42), then we can similarly transform this expression by functions constant on rectangles of the type $\mathbf{1}_J(s) \otimes \mathbf{1}_I(\theta)$, and

we show the continuity of this map.

$$s \to \int_{s-t}^{s} \mathbf{1}_I(\tau) \otimes \mathbf{1}_I(s+\theta-\tau)\zeta d\tau. \tag{49}$$

In the same way as in (47), we us evaluate the difference of the values of (49) at two values $s_1$ and $s_2$ of $s$, denoting by :

$$\begin{aligned}
\|G\| &= \left\| \int_{s_1-t}^{s_1} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_1+\theta-\tau)\zeta d\tau - \int_{s_2-t}^{s_2} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_2+\theta-\tau)\zeta d\tau \right\| \\
&\leq \left\| \int_{s_1-t}^{s_2-t} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_1+\theta-\tau)\zeta d\tau + \int_{s_2}^{s_1} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_1+\theta-\tau)\zeta d\tau \right\| \\
&\quad + \left\| \int_{s_1-t}^{s_2} \mathbf{1}_J(\tau) \otimes (\mathbf{1}_I(s_1+\theta-\tau) - \mathbf{1}_I(s_2+\theta-\tau))\zeta d\tau \right\| \\
&\leq \left\| \int_{s_1-t}^{s_2-t} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_1+\theta-\tau)\zeta d\tau + \int_{s_2}^{s_1} \mathbf{1}_J(\tau) \otimes \mathbf{1}_I(s_1+\theta-\tau)\zeta d\tau \right\| \\
&\quad +4\,|s_1-s_2|\,\|\zeta\|\,.
\end{aligned}$$

Thus $s \to \int_0^t \phi(s-\tau)(\tau+\theta)d\tau$, is continuous, and also

$$\left.\begin{array}{l} s \to \int_0^t (\int_0^{\max(0,\bullet)} L(u_\sigma(s-\tau)d\sigma)_\tau(\theta)d\tau \\ s \to \int_0^{-\theta} \phi(s-\tau)(\tau+\theta)d\tau \end{array}\right\} \text{are continuous.}$$

Finally $s \to S_L(t)(\phi,x)(s)$ is continuous.
For the continuity in $t$, we have

$$(S_L(t+h)f)(s)(\theta) - (S_L(t)f)(s)(\theta) = \int_t^{t+h} T_L(\tau)f(s-\tau)(\theta)d\tau.$$

From (25), $T_L$ is exponentially bounded, and,

$$\|(S_L(t+h)f)(s) - (S_L(t)f)(s)\| \leq \int_t^{t+h} \|T_L(\tau)f(s-\tau)\|\,d\tau.$$

Therefore

$$\begin{aligned}
\|S_L(t+h)f - S_L(t)f\| &\leq (\int_t^{t+h} Me^{\tau\omega}d\tau)\,\|f\| \\
&\leq \frac{M}{\omega}e^{t\omega}(e^{h\omega}-1)\,\|f\|
\end{aligned}$$

and consequently $t \to S_L(t)(\phi,x)$ is continuous from $\mathbb{R}^+$ into $BR(\mathbb{R},R_X^+)$, which entails the strong continuity of $(S_L(t))_{t\geq 0}$ in $BR(\mathbb{R},R_X^+)$.□

**Theorem 23** *$S(t)$ defined by formula (41) is a norm continuous integrated semigroup on $BR(\mathbb{R},R_X)$.*

**Proof** : Let $f \in BR(\mathbb{R},R_X^+)$, in the same way as in Theorem 15 we can prove that

$$\int_0^s (S(t+\sigma) - S(\sigma))f)(\eta)d\sigma = (S(s)S(t)f)(\eta)$$

then

$$S(s)S(t) = \int_0^s (S(t+\sigma) - S(\sigma))d\sigma, \tag{50}$$

and

$$S(0) = 0.$$

Consequently, $S(t)$ is an integrated semigroup on $BR(\mathbb{R}, R_X^+)$.
Also

$$S(t+h)f - S(t)f = \int_t^{t+h} T(\tau)f d\tau.$$

By using (36), we have ,

$$\|T(\tau)f\| \leq CM_{t+1}e^{(t+1)\omega}\|f\|, \ \tau \in [t, t+h]$$

where $M_t = \sup_s (\prod_{s-t<t_p<s} \left\| \widetilde{B}_{(t_p)} \right\|)$, and $C$ is a constant.
We then see that

$$\|S(t+h)f - S(t)f\| \leq CM_{t+1}e^{(t+1)\omega}h\|f\|$$

which shows the norm continuity of the integrated semigroup $S(t)$. $\square$

We are now in position to prove the first main result of this work, namely, Theorem 1 (stated in the Introduction):

**Proof of Theorem 1:** Let $f \in BR(\mathbb{R}, R_X^+)$, and we suppose that $0 \leq t < \delta$

$$(S(t)f)(s)(\theta) = \int_0^t (T(\tau)f)(s)(\theta)d\tau.$$

There are 2 cases to discuss:

$a$) If $t_i < s < t_i + \delta$ and $s - t \leq t_i$, then, we have

$$\begin{aligned}
(S(t)f)(s)(\theta) &= \int_0^{s-t_i} (T(\tau)f)(s)(\theta)d\tau + \int_{s-t_i}^t (T(\tau)f)(s)(\theta)d\tau \\
&= \int_0^{s-t_i} (T_L(\tau)f(s-\tau))(\theta)d\tau + \\
&\quad \int_{s-t_i}^t (T_L(s-t_i) \circ \widetilde{B}_i \circ T_L(t_i - (s-\tau))f(s-\tau))(\theta)d\tau \\
&= (S_L(s-t_i)f)(s)(\theta) + \\
&\quad (T_L(s-t_i) \circ \widetilde{B}_i \circ \int_0^{t-s+t_i} T_L(\tau)f(t_i-\tau))(\theta)d\tau \\
&= (S_L(s-t_i)f)(s)(\theta) + \\
&\quad (T_L(s-t_i) \circ \widetilde{B}_i \circ S_L(t-s+t_i)f(t_i))(\theta).
\end{aligned}$$

$b$) If $s < t_{i+1}$ and $s - t > t_i$ : thus

$$\begin{aligned}
(S(t)f)(s) &= \int_0^t (T(\tau)f)(s)d\tau \\
&= \int_0^t T_L(\tau)f(s-t)d\tau.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s)(\theta) &= \widetilde{B}_i(\int_0^t T_L(\tau)f(t_i-\tau)(\theta)d\tau) \\
&= (\int_0^t T_L(\tau)f(t_i-\tau)(\theta)d\tau, B_i(\int_0^t T_L(\tau)f(t_i-\tau)(0)d\tau))
\end{aligned}$$

and

$$\lim_{\substack{s \to t_i \\ <}} (S(t)f)(s)(\theta) = \int_0^t (T_L(\tau) f(t_i - \tau))(\theta) d\tau \in R_X.$$

Also, $(S(t)f)$ remains discontinuous at points $s = t_i$. On the other hand, $S(t)f$ is continuous at all other points $s \notin \{t_i\}_{i \in \mathbb{Z}}$.
By using the same technique as a Theorem 16, we have the result for all $t \geq 0$.
If we note by

$$\Phi(\theta) = \int_0^t (T_L(\tau) f(t_i - \tau))(\theta) d\tau,$$

then

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s) = (\Phi, B_i[\Phi(0)]).$$

The map is continuous at all points $s \notin \{t_i\}$, thus singularities are concentrated in $\{t_i\}$ :

$$\lim_{\substack{s \to t_i \\ >}} (S(t)f)(s)(0) = B_i \left[ \lim_{\substack{s \to t_i \\ <}} (S(t)f)(s)(0) \right].$$

Thus the proof of Theorem 1 is complete.□

We also remark that the integration of the semigroup made it possible to eliminate all discontinuities due to the data, except at points $s = t_i$, $i \in \mathbb{Z}$.

## 6 Generator and domain of the integrated semigroup

We denote the set of first kind singularities of a function $f$, (i.e the points of discontinuity of $f$ on $\mathbb{R}$) by $\mathrm{sing}(f)$.
We will now show our main result, Theorem 2, described in Introduction, stating that the operator $\mathcal{A}$ defined hereafter is the infinitesimal generator of the continuous integrated semigroup $S(t)$ :

$$\begin{aligned} (\mathcal{A}f)(s) &= \frac{\partial}{\partial \theta} f(s)(\theta) - \frac{\partial}{\partial s} f(s)(\theta),\ s \notin \{t_i\} \\ (\mathcal{A}f)(s)(0) &= \frac{\partial f}{\partial \theta}(s) - f^{'}(s)(0) \end{aligned} \tag{51}$$

$$D(\mathcal{A}) = \left\{ \begin{array}{l} f \in BR(\mathbb{R}, R_X^+) :\ \mathrm{sing} f \subset \{t_i\}_{i \in \mathbb{Z}},\ \lim\limits_{\substack{s \to t_i \\ >}} f(s) = \lim\limits_{\substack{s \to t_i \\ <}} \widetilde{B}_i f(s), \\ \text{and } \dfrac{\partial f(s)}{\partial \theta}(\theta) \in R_X^+,\ \dfrac{\partial f}{\partial s} \in BR(\mathbb{R}, R_X^+),\ \mathrm{sing} \dfrac{\partial f}{\partial s} \subset \{t_i\}_{i \in \mathbb{Z}}, \\ \dfrac{\partial f}{\partial \theta}(s)(0) = Lf(s),\ s \notin \{t_i\}_{i \in \mathbb{Z}}. \end{array} \right\}.$$

Formula (51) may be obtained as follows, if $s \notin \{t_i\}_{i \in \mathbb{Z}}$, and $t > 0$ is small enough, we have

$$(T(t)f)(s) = T_L(t) f(s - t)$$

so that

$$\frac{(T(t)f)(s) - f(s)}{t} = \left\{ \frac{T_L(t) - I}{t} \right\} f(s - t) + \frac{f(s - t) - f(s)}{t}.$$

Thus, as $t \to 0$, we have

$$\begin{aligned}(\mathcal{A}f)(s) &= \frac{\partial f}{\partial \theta}(s) - f^{'}(s),\\(\mathcal{A}f)(s)(0) &= \frac{\partial f}{\partial \theta}(s) - f^{'}(s)(0).\end{aligned}$$

**Proof of Theorem 2:** We will show that $\mathcal{A}$ is a Hille-Yosida **(HY)** operator. For this, we first have to determine the resolvent operator:

$$R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}.$$

Given $f \in BR(\mathbb{R}, R_X^+)$,
the equation reads as (with $v \in D(\mathcal{A})$)

$$\begin{aligned}\lambda v(s) - \mathcal{A}v(s) &= f(s)\\\lambda v(s)(\theta) - \frac{\partial}{\partial \theta}v(s)(\theta) + \frac{\partial}{\partial s}v(s)(\theta) &= f(s)(\theta).\end{aligned} \tag{52}$$

We calculate $v$ by the method of lines. We denote

$$\widetilde{v}(\sigma) = v(a-\sigma)(\sigma)$$

where $a$ is any constant. Then, from (52)

$$\begin{aligned}\lambda \widetilde{v}(\sigma) - \widetilde{v}^{'}(\sigma) &= f(a-\sigma)(\sigma)\\(e^{-\lambda\sigma}\widetilde{v}(\sigma))^{'} &= -e^{-\lambda\sigma} f(a-\sigma)(\sigma),\end{aligned}$$

and by integration

$$\begin{aligned}e^{-\lambda\sigma}\widetilde{v}(\sigma) - \widetilde{v}(0) &= -\int_0^{\sigma} e^{-\lambda t} f(a-t)(t)dt\\\widetilde{v}(\sigma) &= e^{\lambda\sigma}\widetilde{v}(0) - \int_0^{\sigma} e^{\lambda(\sigma-t)} f(a-t)(t)dt\\v(a-\sigma)(\sigma) &= e^{\lambda\sigma} v(a)(0) + \int_{\sigma}^{0} e^{\lambda(\sigma-t)} f(a-t)(t)dt.\end{aligned}$$

Thus

$$v(s)(\theta) = e^{\lambda\theta} v(s+\theta)(0) + \int_{\theta}^{0} e^{\lambda(\theta-t)} f(s+\theta-t)(t)dt, \tag{53}$$

for $s \in \mathbb{R}$, $\theta \in [-r, 0]$, consequently,

$$\begin{aligned}|v(s)(\theta)| &\leq e^{\lambda\theta}|v(s+\theta)(0)| + \frac{1}{\lambda}(1-e^{\lambda\theta})\|f\|\\&\leq e^{\lambda\theta}(|v(s+\theta)(0)| - \frac{1}{\lambda}\|f\|) + \frac{1}{\lambda}\|f\|\\&\leq \max(\sup_{\theta\in[-r,0]}|v(s+\theta)(0)|, \frac{1}{\lambda}\|f\|).\end{aligned} \tag{54}$$

To estimate $v(s)(\theta)$, it is first necessary to estimation $v(s)(0)$.
From (52), and (53) we have

$$\frac{\partial}{\partial \theta}v(s)(0) = \lambda v(s)(0) + \frac{\partial}{\partial s}v(s)(0) - f(s)(0).$$

In view of the definition of $D(\mathcal{A})$, we have $\dfrac{\partial}{\partial\theta}v(s) = L(v(s))$, which yields

$$\lambda v(s)(0) + \frac{\partial}{\partial s}v(s)(0) = L(e^{\lambda\bullet}v(s+\bullet)(0)) + L\{\int_{\bullet}^{0} e^{\lambda(\bullet - t)} f(s+\bullet-t)(t)dt\} + f(s)(0).$$

We denote

$$v(s)(0) = W(s),$$

In terms of $W$, the equation reads

$$\begin{aligned} \lambda W(s) + W^{'}(s) &= L(e^{\lambda\theta}W_s) + \\ & \quad L\{\int_{\theta}^{0} e^{\lambda(\theta-t)} f(s+\theta-t)(t)dt\} + f(s)(0), \qquad (55) \\ W(t_i^+) &= B_i W(t_i^-). \end{aligned}$$

(This condition comes from the definition of $D(\mathcal{A})$). We multiply both sides of (55) by $e^{\lambda s}$, and we define:

$$Z(s) = e^{\lambda s}W(s),\ g(s) = e^{\lambda s}L\{\int_{\bullet}^{0} e^{\lambda(\bullet - t)} f(s+\bullet-t)(t)dt\} + e^{\lambda s}f(s)(0). \qquad (56)$$

We are looking for $v \in BR(\mathbb{R}, X)$, we should have $W \in BR(\mathbb{R}, R_X^+)$. On the other hand, $f \in BR(\mathbb{R}, R_X^+)$, we defined the space

$$\mathcal{Z}_\lambda = \left\{ z : |z|_\lambda \underset{\mathrm{def}}{=} \sup e^{-\lambda s}\,|z(s)| < +\infty \right\} \qquad (57)$$

and $|z|_\lambda$ is the norm of this space, for all $\lambda > 0$. By using (56) we can shows that

$$|Z|_\lambda = \sup_{s\in\mathbb{R}} e^{-\lambda s}\,|Z(s)| < \infty \qquad (58)$$

and

$$|g|_\lambda = \sup_{s\in\mathbb{R}} e^{-\lambda s}\,|g(s)| < M, \qquad (59)$$

The equation in $W$ becomes

$$\begin{aligned} Z^{'}(s) &= L(Z_s) + g(s),\ \forall s \in \mathbb{R} \qquad (60) \\ Z(t_i^+) &= B_i Z(t_i). \qquad (61) \end{aligned}$$

We first solve equation (60) **without impulses.** For that, integrating (60), from $-\infty$ to $s$, and denoting $\widetilde{Z}$ the solution, we obtain:

$$\widetilde{Z}(s) = \int_{-\infty}^{s} L(\widetilde{Z}_u)du + \int_{-\infty}^{s} g(u)du. \qquad (62)$$

From (58), we have

$$\left|\widetilde{Z}_u\right| \le e^{\lambda u}\left|\widetilde{Z}\right|_\lambda,\ \forall u.$$

We define an operator

$$(\mathcal{K}\widetilde{Z})(s) = \int_{-\infty}^{s} L(\widetilde{Z}_u)du,$$

in the space $\mathcal{Z}_\lambda$ defined in (57), then

$$\begin{array}{rcl} \left|\mathcal{K}\widetilde{Z}\right|_\lambda & = & \sup_s e^{-\lambda s}\left|(\mathcal{K}\widetilde{Z})(s)\right| \\ & \leq & \|L\|\, e^{-\lambda s}\int_{-\infty}^{s} e^{\lambda u}\left|\widetilde{Z}\right|_\lambda du \\ & \leq & \dfrac{\|L\|}{\lambda}\left|\widetilde{Z}\right|_\lambda \end{array}$$

and we see that

$$|\mathcal{K}|_\lambda \leq \frac{\|L\|}{\lambda}$$

which is a strict contraction, when $\lambda$ is large enough. Moreover, it holds that $g \in \mathcal{Z}_\lambda \Rightarrow \int_{-\infty}^{s} g \in \mathcal{Z}_\lambda$, and we have :

$$\left|\int_{-\infty}^{\bullet} g\right|_\lambda \leq \frac{1}{\lambda}|g|_\lambda$$

Therefore, for each $t$ the equation (60) without impulses in $\widetilde{Z}$ has one and only one solution $\widetilde{Z}$ and verifies the estimation

$$\left|\widetilde{Z}\right|_\lambda \leq \frac{|g|_\lambda}{\lambda - M}. \tag{63}$$

Let us return to the system of equations (60)-(61)

$$\begin{array}{rcl} Z^{'}(s) & = & L(Z_s) + g(s),\ \forall s \in \mathbb{R} \\ Z(t_i^+) & = & B_i Z(t_i^-),\ i \in \mathbb{Z}. \end{array}$$

To order to solve this system, we carry out a change of variable, by replacing $Z$ by $U + \widetilde{Z}$, where $\widetilde{Z}$ is the solution we have just computed.
Thus, we have

$$U^{'}(s) = L(U_s),\ \forall s \in \mathbb{R} \tag{64}$$

and $\widetilde{Z}(t_i^+) = \widetilde{Z}(t_i^-) = \widetilde{Z}(t_i)$, implies

$$U(t_i^+) - B_i U(t_i^-) = [B_i - Id]\,\widetilde{Z}(t_i),\ i \in \mathbb{Z}. \tag{65}$$

We will determine $U$ as a limit of a sequence $U_n$, the $U_n$ being solutions of the following equations:

$$\left\{\begin{array}{lcl} U_n^{'}(s) & = & L(U_n)_s,\ \forall s > t_{-n} \\ U_n(s) & = & 0,\ s \leq t_{-n} \\ U_n(t_{-n}^+) & = & [B_{-n} - Id]\,\widetilde{Z}(t_{-n}) \\ U_n(t_i^+) - B_i U_n(t_i) & = & [B_i - Id]\,\widetilde{Z}(t_i),\ i \geq -n+1,\ i \in \mathbb{Z}. \end{array}\right. \tag{66}$$

If we denote by

$$u_n(s) = U_{n+1}(s) - U_n(s).$$

Then, we have

$$u_n^{'}(s) = L(u_n)_s,\ s > t_{-n}$$

and for $t_{-n-1} < s \le t_{-n}$, we have

$$\begin{aligned} u_n^{'}(s) &= U_{n+1}^{'}(s) - U_n^{'}(s) \\ &= U_{n+1}^{'}(s) \\ &= L(U_{n+1})_s \\ &= L(U_{n+1})_s - L(U_n)_s \\ &= L(u_n)_s, \end{aligned}$$

thus

$$u_n^{'}(s) = L(u_n)_t,\ s > t_{-n-1}.$$

and

$$u_n(t_i^+) - B_i u_n(t_i) = 0,\ i \ge -n+1.$$

Since $U_n(t_{-n}) = 0$,

$$\begin{aligned} u_n(t_{-n}^+) - B_n u_n(t_{-n}) &= U_{n+1}(t_{-n}^+) - U_n(t_{-n}^+) - B_n U_{n+1}(t_{-n}) + B_n U_n(t_{-n}) \\ &= [B_{-n} - Id]\,\widetilde{Z}(t_{-n}) - [B_{-n} - Id]\,\widetilde{Z}(t_{-n}) \\ &= 0 \end{aligned}$$

and also $U_n(t_{-n-1}^+) = 0$, because $t_{-n-1} < t_{-n}$, we see that

$$\begin{aligned} u_n(t_{-n-1}^+) &= U_{n+1}(t_{-n-1}^+) - U_n(t_{-n-1}^+) \\ &= [B_{-n-1} - Id]\,\widetilde{Z}(t_{-n-1}) \end{aligned}$$

and for $s \le t_{-n-1} < t_{-n}$, we have :

$$u_n(s) = 0.$$

Finally $u_n$ verifies the delay differential equation with impulses

$$\begin{cases} {u_n}^{'}(s) &= L(u_n)_s,\ s > t_{-n-1} \\ u_n(t_i^+) - B_i u(t_i) &= 0,\ i \ge -n \\ u_n(t_{-n-1}^+) &= (B_{-n-1} - Id)\widetilde{Z}(t_{-n-1}) \\ u_n(s) &= 0,\ s \le t_{-n-1}. \end{cases}$$

It follows from (63) that

$$\begin{aligned} \left|\widetilde{Z}(t_{-n-1})\right| &\le \frac{K}{\lambda} e^{\lambda t_{-n-1}} \\ &\le \frac{K}{\lambda} e^{-n\delta\lambda}. \end{aligned}$$

Then, from Lemma 24 (see annex), we have

$$\|u_n(s)\| \le \xi n e^{-n\delta\lambda}$$

when $\xi$ is a constant and $\lambda > 0$ large enough.
It is now clear that $U_n(s)$ is a Cauchy sequence of bounded functions on $\mathbb{R}$.
Then, the sequence $U_n(t)$ converges on $\mathbb{R}$ to a bounded function which is bounded on $\mathbb{R}$, and satisfies equation (66). Thus, the function $U + \widetilde{Z}$ is a solution to the equation (60)-(61). Moreover, it is bounded on $\mathbb{R}$. Thus

$$Z(s+\theta)e^{-(s+\theta)\lambda} = v(s+\theta)(0),$$

and, for $\lambda$ large enough

$$\sup_{\theta\in[-r,0]} |v(s+\theta)(0)| \le \frac{\|f\|}{\lambda}.$$

Then from (54), we have for all $\lambda$ large enough

$$\left|[(\lambda I-\mathcal{A})^{-1}f](s)(\theta)\right| \leq \frac{1}{\lambda-M}\left\|f\right\|.$$

Therefore we see finally that

$$\left\|(\lambda I-\mathcal{A})^{-1}f\right\| \leq \frac{1}{\lambda-M}\left\|f\right\|,$$

thus, $\mathcal{A}$ is a **(HY)** operator, therefore it determines an locally Lipschitz continuous integrated semigroup $(S(t)f)$ on $BR(\mathbb{R}, R_X^+)$.$\square$

## 7 Conclusion

We have shown in this work that to each of a large class of delay differential equations with impulses one can associate an integrated semigroup in the space $BR(\mathbb{R}, R_X)$ and we have determined the infinitesimal generator of this integrated semigroup. From the theory of integrated semigroup (Lemma 5 and Theorem 7), see the literature [1], [2], [3], [4], [5], [6], [7], it is known that $T(t)_{|\overline{D(\mathcal{A})}}$, $t \geq 0$, constitutes a $C_0$-semigroup on $\overline{D(\mathcal{A})}$, with the same infinitesimal generator $\mathcal{A}$. Conversely, it is possible to return from the semigroup to the evolution operator solution of the equation with impulses. What can we gain from such results? Obviously, one can expect to use the body of results that have been developed for semigroups to the case of equations with impulses. Examples of that possibility include the treatment of semilinear equations, using a variation of constant formula, or the computation of the spectrum of the generator as a first step in the quest for stability. If we consider a equation with implicit impulses, the study of stability of a steady state solution may easily benefit from such a theory. Such issues will be considered in future work.

## Appendix A -Glossary-

| | | |
|---|---|---|
| $BR(\mathbb{R}, X)$ | : | space of bounded regulated functions continuous on the left from $\mathbb{R}$ into $X$ |
| $BC(\mathbb{R}, X)$ | : | space of bounded continuous functions from $\mathbb{R}$ into $X$ |
| $BUC(\mathbb{R}, X)$ | : | space of bounded uniformly continuous functions from $\mathbb{R}$ into $X$ |
| $C_X$ | : | space of continous function on $[-r, 0]$ to $X$. |
| $R_X$ | : | space of regulated function continuous to the left on $[-r, 0]$ to $X$. |
| $R_X^+$ | : | $R_X^+ \times X$ -extension of the space $R_X$ at the right of the point 0- |
| $\mathcal{L}(X)$ | : | space of bounded linear operator on $X$ |
| $L$ | : | linear operator on $C_X$, or $R_X^+$ |
| $R(\lambda, A)$ | : | resolvent of $A$ in $\lambda$ |
| $D(A)$ | : | domain of $A$ |
| $\overline{D(A)}$ | : | closure of $D(A)$ |
| $A_{\vert Y}$ | : | part of $A$ in $Y$ |
| $\omega_0$ | : | growth bound of a semigroup |
| $(t_i)_{i\in\mathbb{Z}}$ | : | increasing family of real numbers, support of the impulses |
| $\delta$ | : | $\inf\limits_{i\in\mathbb{Z}}(t_{i+1} - t_i)$ |
| $\mathbf{1}_J$ | : | indicator function of the set $J$ |
| $\mathbf{1}_I \otimes \mathbf{1}_J$ | : | indicator function for the rectangle $I \otimes J$ |
| $(T_L(t))_{t\geq 0}$ | : | semigroup of linear operators associated to linear operators $L$ without impulses |
| $\mathcal{U}$ | : | fundamental solution of the linear equation associated to linear operator $L$ without impulses |
| $(T(t))_{t\geq 0}$ | : | semigroup of linear operators with impulses |
| $(T(t)_{\vert Y})_{t\geq 0}$ | : | restriction of $T(t)$ to $Y$ |
| $(S_L(t))_{t\geq 0}$ | : | integrated semigroup of linear operators associated to a linear operator $L$ without impulses |
| $(S(t))_{t\geq 0}$ | : | integrated semigroup of linear operators associated with $(T(t))_{t\geq 0}$ |

## Appendix B -Note-

We consider the delay differential equation with impulses

$$\begin{cases} {u_n}'(t) & = & L(u_n)_t,\ t > t_{-n-1} \\ u_n(t_i^+) - B_i u(t_i) & = & 0,\ i \geq -n \\ u_n(t) & = & 0,\ t \leq t_{-n-1} \\ u_n(t_{-n-1}^+) & = & (B_{-n-1} - Id)\widetilde{Z}(t_{-n-1}) \end{cases} \tag{67}$$

where $\widetilde{Z}$ is a given function and we denote by $\mathcal{U}$ the fundamental solution of the delay differential equation without impulses. $\mathcal{U}$ is exponentially bounded, with a growth $\omega$,

$$\|\mathcal{U}(t-\sigma)\| \leq M e^{\omega(t-\sigma)},\ t \geq \sigma, \tag{68}$$

where $M$ is a constant $\geq 1$.

**Lemma 24** *The solution of the equation (67), is given by*

$$u_n(t) = \sum_{i=-n-1}^{N^*} \mathcal{U}(t-t_i)\beta_i = \sum_{i=-1}^{n+N^*} \mathcal{U}(t-t_{-n+i})\beta_{-n+i}, \tag{69}$$

*where $N^* \in \mathbb{N}$, fixed, and $\mathcal{U}$ is a fundamental solution of (67), and the sequence $\beta_{-n+k}$, $k \geq 0$ is defined by*

$$\begin{aligned} \beta_{-n+k} \quad = \quad & (B_{-n+k} - Id)\left\{\mathcal{U}(t_{-n+k} - t_{-n-1})\right. \\ & +\mathcal{U}(t_{-n+k} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n+1})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n+1})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n}) \\ & (B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & +... \\ & +\mathcal{U}(t_{-n+k} - t_{-n+k-1})(B_{-n+k-1} - Id)\mathcal{U}(t_{-n+k-1} - t_{-n-1}) \\ & +... \\ & +\mathcal{U}(t_{-n+k} - t_{-n+k-1})(B_{-n+k-1} - Id)\mathcal{U}(t_{-n+k-1} - t_{-n+k-2}) \\ & \left. ...(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1})\right\}\beta_{-n-1}. \end{aligned}$$

*Moreover,*

$$\left\|\widetilde{Z}(t_{-n-1})\right\| \leq e^{\mu t_{-n-1}}, \tag{70}$$

*when $\mu \in \mathbb{R}$ is large, and $(B_i)_{i\in\mathbb{Z}}$ is a family of uniformly bounded linear operators*

$$\sup_{i\in\mathbb{Z}} \|B_i\| < \infty. \tag{71}$$

*Thus*

$$\|u_n(t)\| \leq K(n + N^* + 2)e^{n(\ln(C+1) - \delta(\mu-\omega))},$$

*where $K$ and $C$ are constants independent of $n$.*

**Proof :** for all $t \in \left]t_{-n-1}, t_{-n}\right]$, we can write the solution $u_n(t)$, of equation (67), as

$$u_n(t) = \mathcal{U}(t - t_{-n-1})\beta_{-n-1},$$

where

$$\beta_{-n-1} = u_n(t_{-n-1}^+) = (B_{-n-1} - Id)\widetilde{Z}(t_{-n-1}).$$

Also, for $t \in ]t_{-n}, t_{-n+1}]$, we have

$$u_n(t) = \mathcal{U}(t - t_{-n-1})\beta_{-n-1} + \mathcal{U}(t - t_{-n})\beta_{-n}.$$

Then,

$$\beta_{-n} = u_n(t_{-n}^+) - \mathcal{U}(t_{-n} - t_{-n-1})\beta_{-n-1},$$

and we have

$$\begin{aligned} u_n(t_{-n}^+) &= B_{-n}u(t_{-n}) \\ &= B_{-n}\mathcal{U}(t_{-n} - t_{-n-1})\beta_{-n-1}. \end{aligned}$$

Thus

$$\begin{aligned} \beta_{-n} &= B_{-n}\mathcal{U}(t_{-n} - t_{-n-1})\beta_{-n-1} - \mathcal{U}(t_{-n} - t_{-n-1})\beta_{-n-1} \\ &= (B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1})\beta_{-n-1}. \end{aligned}$$

Now, we prove the result by induction. For this, we suppose that the result is true for $\beta_{-n+k-1}$. That is

$$\begin{aligned} \beta_{-n+k-1} = \; & (B_{-n+k-1} - Id)\{\mathcal{U}(t_{-n+k-1} - t_{-n-1}) \\ & + \mathcal{U}(t_{-n+k-1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & + \mathcal{U}(t_{-n+k-1} - t_{-n+1})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n-1}) \\ & + \mathcal{U}(t_{-n+k-1} - t_{-n+1})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & + ... \\ & + \mathcal{U}(t_{-n+k-1} - t_{-n+k-2})(B_{-n+k-1} - Id)\mathcal{U}(t_{-n+k-1} - t_{-n-1}) \\ & + ... \\ & + \mathcal{U}(t_{-n+k-1} - t_{-n+k-2})(B_{-n+k-2} - Id)\mathcal{U}(t_{-n+k-2} - t_{-n+k-3}) \\ & ...(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1})\}\beta_{-n-1}, \end{aligned}$$

and one then, we shows that also $\beta_{-n+k}$, satisfies the inequality.
Let $t \in ]t_{-n+k}, t_{-n+k+1}]$, we have

$$u_n(t) = \sum_{i=-n-1}^{-n+k} \mathcal{U}(t - t_i)\beta_i = \sum_{i=-1}^{k} \mathcal{U}(t - t_{-n+i})\beta_{-n+i}.$$

Then

$$\begin{aligned} u_n(t_{-n+k}^+) &= \sum_{i=-1}^{k} \mathcal{U}(t_{-n+k} - t_{-n+i})\beta_{-n+i} \\ &= \sum_{i=-1}^{k-1} \mathcal{U}(t_{-n+k} - t_{-n+i})\beta_{-n+i} + \beta_{-n+k} \end{aligned}$$

and one has

$$\begin{aligned} u_n(t_{-n+k}^+) &= B_{-n+k}u(t_{-n+k}) \\ &= B_{-n+k}\sum_{i=-1}^{k-1} \mathcal{U}(t_{-n+k} - t_{-n+i})\beta_{-n+i}. \end{aligned}$$

As a consequence

$$\begin{aligned}\beta_{-n+k} &= (B_{-n+k} - Id)\left\{\sum_{i=-1}^{k-1} \mathcal{U}(t_{-n+k} - t_{-n+i})\beta_{-n+i}\right\} \\ &= (B_{-n+k} - Id)\left\{\sum_{i=-1}^{k-1} \mathcal{U}(t_{-n+k} - t_{-n+i})\beta_{-n+i}\right\}.\end{aligned}$$

Proceeding, step by step

$$\begin{aligned}\beta_{-n+k} = \; & (B_{-n+k} - Id)\{\mathcal{U}(t_{-n+k} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n+1})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n+2})(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1}) \\ & +\mathcal{U}(t_{-n+k} - t_{-n+2})(B_{-n+2} - Id)\mathcal{U}(t_{-n+2} - t_{-n-1}) \\ & ..... \\ & +\mathcal{U}(t_{-n+k} - t_{-n+k-1})(B_{-n+k-1} - Id)\mathcal{U}(t_{-n+k-1} - t_{-n-1}) \\ & +... \\ & +\mathcal{U}(t_{-n+3} - t_{-n+k-1})(B_{-n+k-1} - Id)\mathcal{U}(t_{-n+k-1} - t_{-n+k-2}) \\ & ...(B_{-n+1} - Id)\mathcal{U}(t_{-n+1} - t_{-n})(B_{-n} - Id)\mathcal{U}(t_{-n} - t_{-n-1})\}\beta_{-n-1}.\end{aligned}$$

From (68), (71) and (70), we have

$$\|\beta_{-n+k}\| \leq MC^2(C+1)^k e^{\omega(t_{-n+k} - t_{-n-1})} e^{\mu t_{-n-1}},$$

where $C$ is a constant.
And also from (69), we can deduce

$$\begin{aligned}\|u_n(t)\| &\leq \sum_{k=-1}^{n+N^*} \|\mathcal{U}(t - t_{-n+k})\| \, \|\beta_{-n+k}\| \\ &\leq M^2C^2(C+1)(n+N^*+2)e^{\omega(t-t_{-n-1})}(C+1)^n e^{\mu t_{-n-1}}.\end{aligned}$$

Consequently for all $t \leq \widetilde{T}$ where $\widetilde{T}$ is a positive constant

$$\begin{aligned}\|u_n(t)\| &\leq M^2C^2(C+1)(n+N^*+2)e^{\omega(t-t_{-n-1})}(C+1)^n e^{\mu t_{-n-1}} \\ &\leq M^2C^2(C+1)e^{\omega\widetilde{T}}(n+N^*+2)e^{(\mu-\omega)t_{-n-1}}e^{n\ln(C+1)}.\end{aligned}$$

As

$$t_{-n-1} \leq -n\delta,$$

it follows that

$$\begin{aligned}\|u_n(t)\| &\leq M^2C^2(C+1)e^{\omega\widetilde{T}}(n+N^*+2)e^{(\mu-\omega)-n\delta}e^{n\ln(C+1)} \\ &\leq M^2C^2(C+1)^{N^*}e^{\omega\widetilde{T}(N^*+1)}(n+N^*+2)e^{n(\ln(C+1)-\delta(\mu-\omega))} \\ &\leq K(n+N^*+2)e^{n(\ln(C+1)-\delta(\mu-\omega))},\end{aligned}$$

where $K = M^2C^2(C+1)^{N^*}e^{\omega\widetilde{T}(N^*+1)}$, is constant.$\square$

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