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A mathematical study of a two-regional population growth model ☆

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Abstract

The paper provides a mathematical study of a model of urban dynamics, adjusting to an ecological model proposed by Lotka and Volterra. The model is a system of two first-order non-linear ordinary differential equations. The study proposed here completes the original proof by using the main tools such as a Lyapunov function. ***To cite this article: J. El Ghordaf et al., C. R. Biologies 327 (2004).***

Résumé

Étude mathématique d'un modèle de croissance de la population de deux régions. Cet article propose un modèle mathématique de dynamique urbaine, ajusté au modèle écologique de Lotka et Volterra. Ce modèle constitue un système de deux équations différentielles ordinaires non linéaires du premier ordre. La présente étude complète la preuve originale par l'utilisation d'outils principaux tels que la fonction de Lyapunov. ***Pour citer cet article : J. El Ghordaf et al., C. R. Biologies 327 (2004).***

Keywords: population growth modelurban dynamics; Lotka–Volterra model; Lyapunov function

Mots-clés : modèle de croissance de population ; dynamique urbaine ; modèle de Lotka–Volterra ; fonction de Lyapunov

1. Introduction

In a world of fast growing population and intercontinental trade relations, the local emergence of new economic centres, on the one hand, and the global competition of existing economic centres, on the other hand, have a major impact on the socioeconomic and

☆ The present work was started with Professor Ovide Arino. Ovide died before finishing this work, we dedicated it to his memory.

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political stability of our future world [1]. So, in the last twenty years, there has been an important change in the way geographers and planners have begun to think about the growth and form of cities. The concept of self-organization by which global structures emerge from local actions has gained popularity [2, 3, etc.] and the reason for the application of this concept is its ability to indicate explicit interdependent relationships between factors of complicated regional systems and to clarify new ordering derived from a fluctuation in this system. Mathematical and computer models have started to flourish. Such models have the ability to capture the essentials of forces' 'attractiveness' and relationships in effect between two cities or between clusters inside cities and to simulate the way these quantities change with time. Analogies with some ecological models are, in some cases, relevant: for example, when two or more species live in proximity and share the same basic requirements, they usually compete for resources, habitat, or territory. Such an analogy has been hardly used in a pioneering model by P. Allen [4].

The Hokkaido prefecture has encoded a serious socio-economic situation, under this problem. Y. Miyata and S. Yamaguchi adapted Allen's model in order to discriminate between a number of roadway systems connecting the main towns with a region [2], so they investigated the possibility of regional redistribution of the population of the Hokkaido prefecture based upon the theory of dynamic self-organization of population distribution. As an illustration of the power of mathematical methods, they consider, in the first part of their work, a simplified system with two towns, adjusting to model proposed by Lotka and Volterra [5], when the competition between two species is depicted without direct reference to the resources they share. The difference of the interpretation between Miyata et al.'s model and the classical Lotka–Volterra model is in the fact that each region has its own potential of resources used by inhabitants of two regions, and the resources are interpreted as services and goods offered in the region. Our purpose here is to revisit this simple system; in the next section, we will describe the model emphasizing its relevance with urban dynamics; in Section 3, we present a complete study of the system, based on the already existing methods in [6] and [7], depending on the location parameters and we will discuss the relevance of the results to the subject completing the mathematical study made by Y. Miyata et al., who used the method of perturbations, which gives no convincing results. The main tools in our study are a Lyapunov function, application of the principle of LaSalle and the discussion of the phase portraits.

2. The model

2.1. Presentation of the model

Miyata et al. present this simple example in order to describe the essence of the theory of dynamic self-organization. So they consider a two-regional population growth model.

The model is represented as follows:

$$\begin{cases} \frac{dx_1}{dt} = k_1x_1(N_1 - x_1 - \beta x_2) - d_1x_1 \\ \frac{dx_2}{dt} = k_2x_2(N_2 - x_2 - \beta x_1) - d_2x_2 \end{cases} \tag{1}$$

where the parameters are defined in the authored paper as follows:

- x_i population of region i;
- N_i carrying capacity of region i;
- k_i birth and immigration rate in region i;
- β parameter of interdependence between the region i and j;
- d_i death and emigration rate in region i.

2.2. Remarks

We have some remarks about interpretation given by the authors concerning the parameters of the model. A better understanding, for us, of the parameters is given as follows:

The general form of an equation describing the evolution of region i, is:

$$\frac{dx_i}{dt} = r_ix_i - d_ix_i$$

where:

- r_ix_i is the growth term;
- d_ix_i is the decrease term;
- r_i is a growth rate, which is not constant and depends on the two population in a symmetric way:

$$r_i = k_i(N_i - x_i - \beta x_j)$$

In the expression of r_i,

- N_i stands for the resources of region i, expressed in terms of the maximum of individuals that the region i can sustain;
- k_i is an efficiency parameter that expresses the interaction between the potential of the region and the population growth;
- β represents the intensity of resources (namely services and goods) offered in a region used by inhabitants of the other regions;
- $N_i - x_i - \beta x_j$ is the part of resources of region i that is unemployed. It represents the potential of the region i, determining the growth of the population x_i through the efficiency parameter k_i.

We suppose, taking the same β in both equations, that an individual divides his consumption in a proportion that is always even between what exists and what is imported. This hypothesis is obviously questionable.

3. Mathematical study of the model

This section is devoted to the asymptotic behaviour of the solution describing the system (1).

System (1) can be written as a classical competition model:

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = k_1 x_1 (P_1 - x_1 - \beta x_2) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = k_2 x_2 (P_2 - x_2 - \beta x_1) \end{cases} \tag{2}$$

where we have introduced the notations

$$P_i = N_i - \frac{d_i}{k_i}, \quad i = 1, 2$$

P_i is the potential of the region i taking into account the mortality rate and the emigration related to this region. Notice that if one of the populations is equal to 0 at some instant, then it is remains equal to 0 at all times $t > 0$. In this case, the equation for the other population is just a logistic that can be written as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = k_i x_i (P_i - x_i), \quad i = 1, 2$$

So, the parameter P_i can be considered as carrying capacities of region i. Then, in order to have $P_i > 0$, we should make the following assumption

Hypothesis. For $i = 1, 2$, the parameters k_i, N_i and d_i satisfy*:*

$$k_i N_i > d_i$$

This hypothesis has an interpretation in the model: $k_i N_i$ is the crude growth rate of population i when the potential is at its maximum value, namely when the potential is almost zero.

The hypothesis means that this maximum crude growth rate is bigger than the mortality and outmigration rate.

Classical theorems apply, providing existence and uniqueness to the initial-value problem associated to (2) with initial value (x_1^0, x_2^0), with $x_i^0 \geqslant 0$, for $i = 1, 2$. Furthermore, the solutions remains non-negative and exist for all $t \geqslant 0$ (see [6]).

3.1. Asymptotic behaviour of the solutions of system (2)

3.1.1. Stationary solutions

The stationary solutions of system (2) satisfy the system:

$$\begin{cases} k_1 x_1 (P_1 - x_1 - \beta x_2) = 0 \\ k_2 x_2 (P_2 - x_2 - \beta x_1) = 0 \end{cases}$$

which gives four equilibrium points:

$$E_0 = (0, 0), \quad E_1 = (P_1, 0), \quad E_2 = (0, P_2) \quad \text{and}$$
$$E_3 = \big((P_1 - \beta P_2)/(1 - \beta^2), (P_2 - \beta P_1)/(1 - \beta^2)\big)$$

Our goal is to study the stability of these points and in the end, we establish the following important result.

3.1.2. Study of the stability

In the addition to existence of the equilibrium points, we can prove that it is only one globally asymptotically stable point in the positive domain.

Lemma 1. *The ω-limit set of each non-negative solution of system (2) is reduced to one equilibrium point.*

Proof. It is a straightforward application of the principle of LaSalle [8], considering the Lyapunov function

$$Q(x_1, x_2) = \tfrac{1}{2} x_1^2 + \beta x_1 x_2 + \tfrac{1}{2} x_2^2 - P_1 x_1 - P_2 x_2$$

(see also [6, p. 270] and [7, p. 28]).

Let us define:

$$V(t) = Q\big(x_1(t), x_2(t)\big)$$

which implies:

$$\dot{V}(t) = -k_1 x_1 (x_1 + \beta x_2 - P_1)^2 - k_2 x_2 (x_2 + \beta x_1 - P_2)^2$$

so

$$\dot{V}(t) \leqslant 0 \quad \text{and}$$
$$\dot{V}(t) = 0 \text{ only at stationary points } E_i, \quad i = 0, 1, 2, 3$$

Since the solutions of the system are bounded, the ω-limit set of each solution is non-empty, compact, connected, and invariant by the system.

By applying the principle of LaSalle, one has:

$$\omega(x) \subset \{E_i, i = 0, 1, 2, 3\}$$

But $\omega(x)$ is a connected set, therefore it is necessarily reduced to one stationary point. So:

$$\omega(x) = \{E_i\} \qquad \square$$

As a consequence of this lemma, each non-negative solution of system (2) approaches asymptotically for $t \to +\infty$ one of the equilibria E_i, $i = 0, \dots, 3$ and in what follows we will locate which of the four equilibria is the limit point.

First of all, let us consider the equilibrium point E_0. The linearized system of (2) around this point is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$$

where

$$x = (x_1, x_2)^\top \quad \text{and} \quad A = \begin{pmatrix} k_1 N_1 - d_1 & 0 \\ 0 & k_2 N_2 - d_2 \end{pmatrix}$$

Since the matrix A has two positive eigenvalues, we can conclude the following lemma:

Lemma 2. *The stationary point $E_0 = (0, 0)$ is repulsive.*

Let us observe that the equilibrium point E_3 is the intersection of the two isoclines

$$R_1 := P_1 - x_1 - \beta x_2 = 0$$
$$R_2 := P_2 - x_2 - \beta x_1 = 0$$

and therefore we are obliged to consider different cases, since for some values of the parameters, R_1 and R_2 do not intersect in IR_+^2.

Due to the symmetry of system (2), we can suppose that $P_2 > P_1$ without any loss of generality.

We will make the study of the asymptotic behaviour of solutions in terms of parameter β, which is significant for the model.

Theorem 3. *If $0 \leqslant \beta < \frac{P_1}{P_2}$, we have:*

(a) *each solution of system (2) corresponding to an initial value $(x_1^0, 0)$, with $x_1^0 > 0$, converges towards the stationary point E_1;*
(b) *each solution of system (2) corresponding to an initial value $(0, x_2^0)$, with $x_2^0 > 0$, converges towards the stationary point E_2;*
(c) *each solution of system (2) corresponding to an initial value (x_1^0, x_2^0), with $x_1^0 > 0$ and $x_2^0 > 0$, converges towards the stationary point E_3.*

Proof. In this case, the isoclines R_1 and R_2 intersect in IR_+^2 and therefore the equilibrium point E_3 should be taken into account.

We obtain the stability of equilibria by linearisation of the system (2) around each equilibrium point E_i, $i = 1, 2, 3$, so that E_1 and E_2 are saddle points and straightforward calculations show that their stable manifolds are $S_1 = \{(x_1, 0), x_1 > 0\}$ and $S_2 = \{(0, x_2), x_2 > 0\}$, respectively, this proves (a) and (b). Furthermore E_3 is asymptotically stable. Bearing in mind Lemma 1 and Lemma 2, assertion (c) holds. $\square$

Moreover, we have the following result:

Lemma 4. *For each $i = 1, 2$, there is a paving of repulsiveness around E_i.*

Proof. With the aim of constructing a paving of repulsiveness around E_1, let $\eta > 0$ be fixed and consider

$$P_\eta = \big\{(x_1, x_2) / |x_1 - P_1| \leqslant \eta,\ 0 \leqslant x_2 \leqslant \eta\big\}$$

Then

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = k_2 x_2 \big(P_2 - \beta P_1 - x_2 - \beta(x_1 - P_1)\big) \geqslant \big(P_2 - \beta P_1 - \eta(\beta + 1)\big) k_2 x_2$$

Choosing

$$0 < \eta \leqslant \frac{P_2 - \beta P_1}{2 + \beta}$$

we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} \geqslant \eta k_2 x_2$$

and then, if a solution is in P_η at some instant t_0, there exists $t_1 > t_0$ for which such solution will be outside P_η for each $t \geqslant t_1$.

In the same way, we can construct a paving of repulsiveness around E_2. □

Theorem 5. *If $\frac{P_1}{P_2} < \beta < \frac{P_2}{P_1}$, we have:*

(a) *each solution of system (2) corresponding to an initial value $(x_1^0, 0)$, with $x_1^0 > 0$, converges towards the stationary point E_1;*
(b) *each solution of system (2) corresponding to an initial value (x_1^0, x_2^0), with $x_1^0 \geqslant 0$ and $x_2^0 > 0$, converges towards the stationary point E_2.*

Proof. In this case, isoclines R_1 and R_2 do not intersect in IR_+^2 and therefore there exist three equilibria E_0, E_1 and E_2. The linearized systems around E_1 and E_2 prove that E_1 is a saddle point with stable manifold S_1 and E_2 is locally asymptotically stable. Bearing in mind Lemma 1, assertion (b) follows. □

Moreover, a paving of repulsivity around E_1 can be constructed in the same way as in Lemma 4.

Finally, for the third case, similar calculations and standard arguments that can be found in [6] lead to:

Theorem 6. *If $\beta > \frac{P_2}{P_1}$, the equilibrium point E_3 is in IR_+^2 and we have:*

(a) *E_0 is unstable;*
(b) *E_1 and E_2 are locally asymptotically stable;*
(c) *E_3 is a saddle point.*

Moreover, the stable manifold associated to E_3 consists of two orbits converging to E_3, which divides the positive quadrant into two basins of attraction Ω_1 and Ω_2. All solutions starting in Ω_1 tend to E_1 and all solutions starting in Ω_2 tend to E_2.

4. Conclusion

We said that the model presented in Section 2 was proposed first by Lotka and Volterra, who described the competition between two species for resources they share rather; it is assumed that the presence of each population leads to a depression of its competitor's growth rate. Later on, this model was adjusted in urban dynamics by changing the meaning of the model's variables and parameters. It is now necessary to make an urban interpretation of the results obtained so far.

Hypothesis $P_1 < P_2$ means that region 2 still the region that has the bigger potential, taking into account the mortality and emigration rate in this region.

When the intensity of resources used by inhabitants of two regions is less strong, $0 \leqslant \beta < \frac{P_1}{P_2}$, we have an interior steady state, which is stable and which means that each population manages to live in its region and to benefit from the services offered in the other region. But when the intensity is much stronger, $\frac{P_1}{P_2} < \beta < \frac{P_2}{P_1}$, then the solution with initial value (x_1^0, x_2^0), with $x_1^0 \geqslant 0$ and $x_2^0 > 0$ converges towards the stationary point E_2. That is the region with the higher potential will be dominant for any initial situation.

For $\frac{P_2}{P_1} < \beta$, we have three non-trivial steady state, with two points which are stable. In this case, it is very difficult to verify which one of the populations leaves its area; it depends crucially on the starting advantage each population has. If the initial condition lies in domain Ω_1, then eventually population 2 leaves to region 1, thus the competition for the use of resources eliminated region 2 and the same if the initial condition lies in domain Ω_2.

As stated above, it has been shown that all solutions of the system (1) converge, depending on the intensity of the competition parameter β and the initial value, to one equilibrium point; from urban interpretation, we conclude that a small change in the quantity of the intensity of competition influences the result.

This study is the commencement of a large and complete mathematical analysis of a complex models initiated by P. Allen et al. and Myata et al. These models concern n regions and consider that the potentials of each region are variable in time with respect to the global population and its distribution between regions.

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