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RESEARCH ARTICLE

Estimation of the Rate of Convergence of Semigroups to an Asynchronous Equilibrium

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Abstract

Perturbations leading to an estimate of the rate of convergence to an asynchronous equilibrium of general semigroups with boundary conditions are given. The particular case of translation semigroups is examined. The results obtained are applied to a demographic model.

1. Introduction

The formulation of models from population dynamics in terms of abstract evolution equations via semigroup methods on Banach spaces has had great success in the last decade. The theory for such equations gives efficient tools for the study of the qualitative properties of the models. In particular, it provides rigorous proofs and intuitive understanding of the ultimate behavior of many diverse population problems structured by an internal variable such as age or size.

Roughly speaking, one can say that the first issue raised by the study of such models has been settled, that is; the determination of asynchronous exponential growth. Let us briefly summarize what this means. Denote $(T(t))_{t\geq 0}$ the semigroup associated with a given model, defined on some Banach space X, A its infinitesimal generator. Asynchronous exponential growth (abbreviated to $A.E.G.$) occurs when one can determine a pair (λ_0, v), $\lambda_0 \in \mathbb{R}$, $0 \neq v \in X$, such that for every $x \in X$ there exists $c \in \mathbb{R}$ such that

$$e^{-\lambda_0 t}T(t)x - cv \to 0 \quad \text{as} \quad t \to +\infty.$$

This asymptotic relation means that the structure of the population (the relative proportions of respective age or size classes) stabilizes asymptotically near a fixed distribution given by a suitable normalization of v not depending on the initial value. Once asynchronous growth has been obtained, the next issue is to determine how fast solutions reach the stable distribution, that is, the speed of convergence of $e^{-\lambda_0 t}T(t)$ to a one dimensional projection operator P.

In the case when $(T(t))_{t\geq 0}$ is eventually compact, this amounts to compare the principal eigenvalue λ_0 to the others and in particular to the number $s_1 = \sup\{\mathrm{Re}\lambda;\ \lambda \in \sigma(A) \setminus \{\lambda_0\}\}$. In the general situation, it amounts to find an estimate of the growth rate of the restriction of $T(t)$ to the space $P^{-1}(0)$. Determining the speed of convergence is a difficult problem and of importance in many contexts. It has notably attracted a lot of attention in numerical analysis where estimates have been found in the frame of matrices or discrete semigroups [13], [9], [11], [14]. Some results have been obtained recently [3] in the case of Lotka–von Foerster–Mc Kendrik models.

In this work, we present results for a reasonably general class of translation semigroups given in [6]. A translation semigroup is characterized by a linear functional representing a suitable boundary condition. We call it the symbol of the semigroup.

The main result of the paper is an estimate of $\lambda_0 - s_1$ in terms of the symbol and a nonzero eigenvector of the adjoint of the generator associated with λ_0 (which is related to the adjoint of the symbol). This result is applied to the Lotka–von Foerster–Mc Kendrik vector type model. We now provide a brief description of the content. Section 2 collects a number of known results in the spectral theory of semigroups. The problem of the estimation of the speed of convergence to asynchronous equilibrium is also stated in this section for semigroups on Banach lattices. The remaining sections are devoted to translation semigroups. Section 3 considers the more general situation of a semigroup determined by a perturbation of a boundary condition. It is preparatory for Section 4 which, after some general facts about translation semigroups, concludes with an estimate of $\lambda_0 - s_1$ in terms of the symbol. Finally, Section 5 illustrates the main theorem of Section 4 by the example already mentioned.

2. The case of a general semigroup

Let us first recall some standard definitions and notations (see for instance [15], [12], [5],[4]) and let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X. Its infinitesimal generator A is a linear, closed, in general unbounded operator with dense domain $D(A)$ in X. We denote, as usual, with $\sigma(A)$ the spectrum of A, $\rho(A)$ the resolvent set of A and, for $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A. The number $s(A) = \sup\{\mathrm{Re}\lambda; \lambda \in \sigma(A)\}$ ($s(A) = +\infty$ if $\sigma(A) = \phi$) is the spectral bound of A and $\sigma_0(A) = \{\lambda \in \sigma(A); \mathrm{Re}\lambda = s(A)\}$ is the peripheral spectrum of A.

For any $\lambda \in \sigma(A)$, the general eigenspace $N_\lambda(A)$ is the smallest closed subspace of X containing $\cup_{k=1}^{+\infty} N((\lambda - A)^k)$, $\dim N_\lambda(A)$ is the algebraic multiplicity of λ. Note that if λ has a finite algebraic multiplicity, then necessarily there exists a finite integer $p > 0$ such that $N_\lambda(A) = N((\lambda - A)^p)$. The *essential spectrum* $\sigma_e(A)$ is the set of $\lambda \in \sigma(A)$ such that either λ is a limit point of $\sigma(A)$, or $R(\lambda - A)$ is not closed, or $N_\lambda(A)$ is infinite dimensional.

For a linear bounded operator L, we denote by $r(L) = \sup\{|\lambda|; \lambda \in \sigma(L)\}$ the spectral radius of L. Moreover, $\omega_0(T) = \lim_{t\to+\infty} \frac{1}{t} \ln \|T(t)\|$ is the growth bound of the semigroup $(T(t))_{t\geq 0}$.

Definition 2.1. [17] The semigroup $(T(t))_{t\geq 0}$ is said to have $A.E.G$ with intrinsic growth rate $\lambda_0 \in \mathbb{R}$ if there exists a nonzero finite rank operator P such that $\lim_{t\to+\infty} e^{-\lambda_0 t} T(t) = P$ (in the operator norm topology).

The study of such an asymptotic behavior arises naturally in population dynamics (see [4]). The following proposition establishes a relationship between the $A.E.G.$ of the semigroup and the spectrum of its generator.

Proposition 2.2. *[17] Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup in the Banach space X and A its infinitesimal generator. $(T(t))_{t\geq 0}$ has A.E.G. with intrinsic growth rate $\lambda_0 \in \mathbb{R}$ if and only if $\omega_e(T) < \lambda_0$, $\sigma_0(A) = \{\lambda_0\}$ and λ_0 is a simple pole of $(\lambda - A)^{-1}$.*

If $(T(t))_{t\geq 0}$ has $A.E.G.$ with intrinsic growth rate λ_0, then $\lambda_0 = \omega_0(T) = s(A)$ and $P = B_{-1}$ where B_{-1} is the coefficient of order -1 in the Laurent series of $R(\lambda, A)$ about λ_0, or $B_{-1} = \frac{1}{2\pi i}\int_\Gamma R(\lambda, A)d\lambda$ and Γ is a positively oriented circle of sufficiently small radius such that no other point than λ_0 lies on or inside Γ. Moreover, there exists a direct sum decomposition of X as $X = N \oplus S$ with $N = PX$, $S = (I-P)X$. Since λ_0 is a simple pole of $R(\lambda, A)$, λ_0 is an eigenvalue of A, $AP = PA = \lambda_0 P$ and one has $N = \ker(\lambda_0 - A)$, $S = R(\lambda_0 - A)$; N and S are $T(t)$-invariant. On the other hand, $(\lambda_0 - A)$ is a closed linear operator with dense domain and its range $R(\lambda_0 - A)$ is closed, so that $S = (\ker(\lambda_0 - A^*))^\perp = \{x \in X;\ \langle x^*, x\rangle = 0,\ x^* \in \ker(\lambda_0 - A)\}$.

Let $\widetilde{T}(t)$ be the restriction of $T(t)$ to S. It is known that $(\widetilde{T}(t))_{t\geq 0}$ is a semigroup on S, its infinitesimal generator $\widetilde{A}$ is the restriction of A to S and the spectrum of $\widetilde{A}$ is $\sigma(\widetilde{A}) = \sigma(A) \setminus \{\lambda_0\}$. Note, in particular, that $\omega_0(\widetilde{T}) \geq \omega_e(T)$

Proposition 2.3. *Assume that $(T(t))_{t\geq 0}$ has $A.E.G.$ with intrinsic growth rate λ_0. Let $\widetilde{T}(t)$ be the restriction of $T(t)$ to $S = \left\{x \in X;\ T(t)x = o(e^{\lambda_0 t}),\ t \to +\infty\right\}$ and let $\varepsilon = \lambda_0 - \omega_0(\widetilde{T})$. Then we have $\varepsilon > 0$ and for any $\eta \in]0, \varepsilon[$, there exists M such that*

$$\left\|e^{-\lambda_0 t}T(t) - P\right\| \leq Me^{-\eta t}, \quad t \geq 0.$$

Our goal in this paper is to provide an estimate of the speed of convergence of $e^{-\lambda_0 t}T(t)$ to P. This will be done if we find an estimate of $\varepsilon = \lambda_0 - \omega_0(\widetilde{T})$. To do this, we will construct a semigroup $(T_y(t))_{t\geq 0}$ depending on a parameter y in a suitable space, such that the restriction of $T_y(t)$ to S will be equal to $\widetilde{T}(t)$ the restriction of $T(t)$ to S. The parameter y will be chosen in such a way that $\eta_y = \lambda_0 - \omega_0(T_y)$ is as big as possible in $]0, \varepsilon[$.

The construction of $T_y(t)$ and the choice of y can be made in a general framework. In view of the applications we restrict ourselves to semigroups with boundary conditions.

3. Semigroups with boundary conditions

In this section we deal with the case of semigroups having infinitesimal generators mainly determined by boundary conditions and follow the presentation given by Greiner [8].

Let X and Y be two Banach spaces, A a closed linear operator with a domain $D(A)$ dense in X. In many applications, A will be a differentiation operator with its maximal domain.

The operator $L : X \to Y$ is bounded such that $R(L) = Y$

We denote by A_0 the restriction of A to $\ker L$, and for every bounded linear operator $\Phi : X \to Y$ we define A_Φ by

$$D(A_\Phi) = \{f \in D(A);\ Lf = \Phi f\},\ A_\Phi f = Af \text{ for} f \in D(A_\Phi).$$

Then A_0 is the operator corresponding to $\Phi \equiv 0$.

Our aim is to find an estimate of the speed of convergence of the semigroup $(T_\Phi(t))_{t\geq 0}$ generated by A_Φ to an asynchronous equilibrium (when it exists). As in

Section 2, the rate of convergence will be given by the difference $s(A_\Phi) - s(\widetilde{A}_\Phi)$. Thus, our problem is to find a suitable perturbation of A_Φ leading to an estimation of "the second eigenvalue" $s(\widetilde{A}_\Phi)$ of A_Φ. Before doing so, let us summarize some results from [8] which will be useful in this respect.

Lemma 3.1. *Assume that $\lambda \in \rho(A_0)$. Then,*

i) $D(A) = D(A_0) \oplus \ker(\lambda - A)$

ii) The restriction of L to $\ker(\lambda - A)$ is an isomorphism onto Y.

We will denote $L_\lambda = L_{|\ker(\lambda - A)}$ and L_λ^{-1} its inverse for each $\lambda \in \rho(A_0)$.

The following proposition gives conditions under which A_Φ is the infinitesimal generator of a semigroup $(T_\Phi(t))_{t\geq 0}$.

Proposition 3.2. *[8] Assume that A_0 is the generator of a semigroup $(T_0(t))_{t\geq 0}$.*

i) Suppose that there are constants $\gamma > 0$, $\zeta \in \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$, $\lambda > \zeta$, $\|Lf\| \geq \lambda\gamma \|f\|$ for $f \in \ker(\lambda - A)$. Then, for every bounded linear operator $\Phi : X \to Y$, A_Φ is the infinitesimal generator of a semigroup $(T_\Phi(t))_{t\geq 0}$.

ii) Suppose that $\Phi \circ A_0$ has a continuous extension from X into Y. Then A_Φ is the infinitesimal generator of a semigroup $(T_\Phi(t))_{t\geq 0}$. Moreover, for λ large enough one has

$$T_\Phi(t) - T_0(t) = L_\lambda^{-1}\Phi T_\Phi(t) - T_0(t)L_\lambda^{-1}\Phi + \int_0^t T_0(t-s)L_\lambda^{-1}[\Phi(\lambda - A_\Phi)]T_\Phi(s)ds.$$

In the case when A_0 generates an analytic (compact) semigroup, then so does A_Φ.

¿From now on, we assume that A, L, and Φ are such that the operator A_Φ is the infinitesimal generator of a semigroup $(T_\Phi(t))_{t\geq 0}$ having $A.E.G.$ with intrinsic growth rate λ_0.

Since the operator A_Φ is the restriction of A to the domain $D(A_\Phi)$, its seems natural that the perturbed operator A_y to be used for the estimation of "the second eigenvalue" of A_Φ should also be the restriction of A to a domain $D(A_y) \subset D(A)$. An indication for the choice of $D(A_y)$ is given by the following result.

Proposition 3.3. *[8] If a restriction A_1 of A to the domain $D(A_1) \subset D(A)$ is the generator of a semigroup and $D(A_1) = \ker l$ with $l : D(A) \to Y$ a surjective A-bounded operator, then any other restriction of A which is a generator is obtained by perturbing l additively.*

For our purpose we have $l := L - \Phi$, $D(A_\Phi) = \ker l$. We also have to assume that l is surjective (this will be true if L is surjective and dominates Φ, which is the case in our application).

Assume that $l = L - \Phi$ is surjective. Let x_0^* be a fixed element of $\ker(\lambda_0 - A_\Phi^*)$. For any $y \in Y$ we define the operator A_y by

$$D(A_y) := \{f \in D(A);\ Lf = \Phi f - \langle x_0^*, f\rangle y\},\ A_y f := Af \ \text{ for } \ f \in D(A_y).$$

Setting $\varphi := -x_0^* \otimes y$, we have $D(A_y) = \ker(l - \varphi) = \{f \in D(A);\ lf = \varphi f\}$. We write $l_\lambda := l_{|\ker(\lambda - A)}$ for the restriction of l to $\ker(\lambda - A)$ and l_λ^{-1} for its inverse for $\lambda \in \rho(A_\Phi)$.

Proposition 3.4. *If A_Φ is the infinitesimal generator of a semigroup $(T_\Phi(t))_{t\geq 0}$, then*

i) A_y is the generator of a semigroup $(T_y(t))$ and for λ large enough we have

$$T_y(t) - T_\Phi(t) = l_\lambda^{-1}\varphi T_y(t) - T_\Phi(t) l_\lambda^{-1}\varphi + \int_0^t T_\Phi(t-s) l_\lambda^{-1}[\varphi(\lambda - A_y)] T_y(s) ds.$$

In case A_Φ generates an analytic (compact) semigroup, then so does A_y.

ii) If $(T_\Phi(t))$ is eventually compact, then the same holds for $(T_y(t))$.

Proof. If we show that $\varphi \circ A_\Phi$ has a continuous extension, then i) follows from Proposition 3.2 ii). But this follows since

$$(\varphi \circ A_\Phi) f = -\langle x_0^*, A_\Phi f\rangle y = -\lambda_0 \langle x_0^*, f\rangle y.$$

For part ii), we have from i) for λ large enough

$$T_y(t) = (1 - l_\lambda^{-1}\varphi)^{-1} T_\Phi(t)(1 - l_\lambda^{-1}\varphi) + (1 - l_\lambda^{-1}\varphi)^{-1} \int_0^t T_\Phi(t-s) K(s) ds.$$

with $K(s) := l_\lambda^{-1}\varphi(\lambda - A_y)T_y(s)$. The conclusion follows by the fact that $K(s)$ is an operator of rank one . ■

Lemma 3.5. *Let $\lambda \in \rho(A_\Phi)$ and let $\lambda_y := \lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle$. Then we have*

$$x_0^* \in D(A_y^*) \text{ and } A_y^* x_0^* = \lambda_y x_0^*.$$

Proof. Let $\lambda \in \rho(A_\Phi)$, $y \in Y$, $f \in D(A_y)$ and $g := f + \langle x_0^*, f\rangle l_\lambda^{-1} y$. We have $f \in X$ $g \in D(A_\Phi)$ and

$$\begin{aligned} \langle x_0^*, A_y f\rangle &= \langle x_0^*, A_\Phi g\rangle - \langle x_0^*, f\rangle \left\langle x_0^*, A l_\lambda^{-1} y\right\rangle \\ &= \left\langle A_\Phi^* x_0^*, f + \langle x_0^*, f\rangle l_\lambda^{-1} y\right\rangle - \lambda \langle x_0^*, f\rangle \left\langle x_0^*, l_\lambda^{-1} y\right\rangle \\ &= \lambda_0 \langle x_0^*, f\rangle + (\lambda_0 - \lambda)\langle x_0^*, f\rangle \left\langle x_0^*, l_\lambda^{-1} y\right\rangle \\ &= \left\langle \left(\lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle\right) x_0^*, f\right\rangle, \end{aligned}$$

so $x_0^* \in D(A_y^*)$ and $A_y^* x_0^* = \lambda_y x_0^*$. ■

Proposition 3.6. *Assume that $\langle x_0^*, f\rangle \neq 0$ for all $0 \neq f \in N := \ker(\lambda_0 - A_\Phi)$. Then, for every $y \in Y$,*

$$\sigma(A_y) = (\sigma(A_\Phi) \setminus \{\lambda_0\}) \cup \{\lambda_y\},$$

where $\lambda_y = \lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle$, $\lambda \in \rho(A_\Phi)$.

Proof. 1) Since A_y is obtained from A_Φ by a perturbation of rank one, one has $\sigma_e(A_y) = \sigma_e(A_\Phi)$ ([8], Proposition 3.1). 2) By Lemma 3.5, we have $A_y^* x_0^* = \lambda_y x_0^*$ and so $\lambda_y \in \sigma(A_y)$. 3) If $\lambda \neq \lambda_0$ is in $P\sigma(A_\Phi)$, then for $f \in \ker(\lambda - A_\Phi)$, $\langle x_0^*, A_\Phi f\rangle = \bar{\lambda}\langle x_0^*, f\rangle = \langle A_\Phi^* x_0^*, f\rangle = \lambda_0 \langle x_0^*, f\rangle$ implies that $\langle x_0^*, f\rangle = 0$, so $f \in D(A_y)$, hence $A_y f = \lambda f$ and $\lambda \in P\sigma(A_y)$. In the same way, since $x_0^* \in D(A_y^*)$, and if $\lambda \neq \lambda_y$ is in $P\sigma(A_y)$, λ is also in $P\sigma(A_\Phi)$.

4) To end the proof, we have to show that $\lambda_0 \notin \sigma(A_y)$ unless $\lambda_0 = \lambda_y$. If $\lambda_0 \in \sigma(A_y)$, then, since $\lambda_0 \notin \sigma_e(A_\Phi) = \sigma_e(A_y)$, there exists $f \in D(A_y)$ such that $A_y f = \lambda_0 f$. For $\lambda \in \rho(A_\Phi)$, define $g := f + \langle x_0^*, f\rangle l_\lambda^{-1} y$. Then $g \in D(A_\Phi)$ and

$$\begin{aligned} A_\Phi g &= \lambda_0 f + \lambda \langle x_0^*, f\rangle l_\lambda^{-1} y \\ &= \lambda_0 g + (\lambda_0 - \lambda)\langle x_0^*, f\rangle l_\lambda^{-1} y. \end{aligned}$$

Applying x_0^* on both sides one obtains

$$\langle x_0^*, A_\Phi g\rangle = \lambda_0 \langle x_0^*, g\rangle + (\lambda_0 - \lambda)\langle x_0^*, f\rangle \left\langle x_0^*, l_\lambda^{-1} y\right\rangle.$$

On the other hand,

$$\langle x_0^*, A_\Phi g\rangle = \langle A_\Phi^* x_0^*, g\rangle = \lambda_0 \langle x_0^*, g\rangle,$$

hence $\langle x_0^*, f\rangle \left\langle x_0^*, l_\lambda^{-1} y\right\rangle = 0$. But, $\langle x_0^*, f\rangle \neq 0$ because if $\langle x_0^*, f\rangle = 0$ one has $f = g \in D(A_\Phi)$ and $A_\Phi f = A_y f = \lambda_0 f$ which implies that $f \in N$. This contradicts the assumption $\langle x_0^*, f\rangle \neq 0$ for all $f \in N \setminus \{0\}$. So $\left\langle x_0^*, l_\lambda^{-1} y\right\rangle = 0$, hence $\lambda_0 = \lambda_y$ (in view of the formula defining λ_y). ∎

Proposition 3.7. *Assume that A_Φ is the generator of a semigroup with A.E.G.. Assume, moreover, that y can be chosen so that $s(A_y)$ is an eigenvalue of A_y and that, for some $z \in \ker(s(A_y) - A_y)$, one has $\langle x_0^*, z\rangle \neq 0$. Then, the spectral bound of A_y is*

$$s(A_y) = \lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle,$$

where λ is an element of $\rho(A_\Phi)$.

Proof. For any $z \in \ker(s(A_y) - A_y)$ we have $\langle x_0^*, A_y z\rangle = s(A_y)\langle x_0^*, z\rangle$, but by Lemma 3.5 $\left\langle A_y^* x_0^*, z\right\rangle = \lambda_y \langle x_0^*, z\rangle$. So, if $\langle x_0^*, z\rangle \neq 0$, then $s(A_y) = \lambda_y = \lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle$. ∎

Since λ_y depends on $x_0^* \in \ker(\lambda_0 - A_\Phi^*)$ and $l_\lambda^{-1} y \in \ker(\lambda - A)$, we cannot obtain an explicit expression of $s(A_y)$ without more information on the operators A, L and Φ.

In the sequel, we will consider the case when the semigroup $(T_y(t))_{t\geq 0}$ is positive in order to use spectral properties of such operators. Since it is easier to check the positivity of the resolvent of the generator, we will give now an expression of $R(\lambda, A_y)$ in terms of $R(\lambda, A_0)$.

Proposition 3.8. *Let $\Psi = \Phi - x_0^* \otimes y$ and $\Psi_\lambda = \Psi L_\lambda^{-1}$, $\lambda \in \rho(A_0)$. If $\|\Psi_\lambda\| < 1$ for λ large enough , then*

$$R(\lambda, A_y) = R(\lambda, A_0) + L_\lambda^{-1} R(1, \Psi_\lambda)\Psi R(\lambda, A_0).$$

Remark. The fact that $\|\Psi_\lambda\| < 1$ for λ large enough follows from conditions ensuring that A_y is a generator (see Proposition 3.2 and [8]).

4. Translation semigroups

We will now restrict our attention to translation semigroups. In [6], it is proved that the generators of translation semigroups on $X = L^1(]-r,0[\,,Y)$, where Y is an arbitrary Banach space, are first derivative operators, their domain is a part of $W^{1,1}(]-r,0[\,,Y)$ satisfying $f(0) = \Phi f$ where $\Phi : W^{1,1}(]-r,0[\,,Y) \longrightarrow Y$ is a bounded linear operator. Using this fact, a number of qualitative properties of translation semigroups are established in [6]. In [1] a generalization to $X = L^P(]-r,0[\,,Y)$, $1 \leq p < \infty$, is obtained. Compactness of the translation semigroup $T_\Phi(t)$ received a particular attention in that paper and conditions ensuring compactness of $T_\Phi(t)$ as a consequence of properties of Φ are given.

With the above assumptions on Y and X, $0 < r \leq +\infty$, $1 \leq p < +\infty$, and X endowed with the norm $\|f\| = \left(\int_{-r}^{0} \|f\|_Y^p\right)^{1/p}$, a semigroup $(T(t))_{t\geq 0}$ is called a translation semigroup if for all $f \in W^{1,p}(]-r,0[\,,Y)$, $t \geq 0$ and almost every $x \in]-r,0[$ $(T(t)f)(x) = \begin{cases} f(t+x) & \text{if } t+x<0 \\ (T(t+x)f)(0) & \text{if } t+x>0. \end{cases}$

Proposition 4.1. *[6] Let $(T(t))_{t\geq 0}$ be a translation semigroup of linear operators on the Banach space $X = L^p(]-r,0[\,,Y)$. Then, the infinitesimal generator A of $(T(t))_{t\geq 0}$ is such that: i) $D(A) \subset W^{1,p}(]-r,0[\,,Y)$ and $Af = f'$ for all $f \in D(A)$. ii) The map $f \longmapsto f(0)$ is continuous from $(D(A)\,\|.\|_A)$ into Y.*

So, translation semigroups on X are a subclass of semigroups with boundary conditions, for which the operator A is the first derivative and the operator L (see Section 3) is given by $Lf = f(0)$.

Let Φ be a bounded linear operator from X into Y. We define $(T_\Phi(t))_{t\geq 0}$ as the semigroup of linear operators generated by the operator A_Φ given by

$$D(A_\Phi) := \left\{f \in W^{1,p}\,(]-r,0[\,,Y)\,; f(0) = \Phi f\right\}; \quad A_\Phi f := f' \text{ for} f \in D(A_\Phi).$$

Then $(T_\Phi(t))$ is a translation semigroup (see [6], [2]).

We observed in Section 3 that the estimation of the rate of convergence of $e^{-\lambda_0 t}T_\Phi(t)$ to P needs the determination of an $x_0^* \in \ker(\lambda_0 - A_\Phi^*)$ and $l_\lambda^{-1}y$, $y \in Y$, $\lambda \in \rho(A_\Phi)$. For the translation semigroup $T_\Phi(t)$ defined on X, assuming that the dual space of X is $X^* = L^\infty((-r,0),Y^*)$ if $p=1$ (this is the case if Y is reflexive) and $X^* = L^q((-r,0),Y^*)$ where $\frac{1}{p}+\frac{1}{q} = 1$ if $p>1$ (this is the case if Y is separable), we give in the following propositions explicit expressions for x_0^*, $l_\lambda^{-1}y$ and $\lambda_y = \lambda_0 - (\lambda-\lambda_0)\left\langle x_0^*, l_\lambda^{-1}y\right\rangle$, which is the candidate for the estimate of the rate of convergence.

We will use the notations $\varepsilon_\lambda(a) := e^{\lambda a}$, $(\varepsilon_{\lambda\otimes}z)(a) := e^{\lambda a}z$ and $\Phi_\lambda(z) := \Phi(\varepsilon_{\lambda\otimes}z)$ for $z \in Y$

Proposition 4.2. *Let A_Φ be the operator on $X = L^p(]-r,0[\,,Y)$ defined by $D(A_\Phi) = \{f \in W^{1,p}(]-r,0[\,,\ Y);\ f(0) = \Phi f\}$ and $A_\Phi f = f'$ for all $f \in D(A_\Phi)$. Let A_Φ^* be the adjoint operator of A_Φ. Then, $x^* \in X^*$ is a non trivial solution of $A_\Phi^* x^* = \lambda x^*$ if and only if*

$$x^*(a) = e^{-\lambda a}\int_{-r}^{a} e^{\lambda s}[\Phi^* x^*(0)](s)ds \text{ with } x^*(0) = \Phi_\lambda^*[x^*(0)], \quad -r < a < 0$$

Proof. We proceed in two steps. In the first one, we show that if $x^* \in X^*$ is a non trivial solution of $A_\Phi^* x^* = \lambda x^*$ then x^* is differentiable. In the second one, we establish the announced formula.

1) For any $f \in D(A_\Phi)$ we have

$$\begin{aligned} \langle A_\Phi^* x^*, f \rangle &= \langle x^*, A_\Phi f \rangle \\ &= \langle x^*, f' - \mu f \rangle + \mu \langle x^*, f \rangle \quad \text{for all } \mu \in I\!R. \end{aligned}$$

So, $\langle A_\Phi^* x^*, f \rangle = \lambda \langle x^*, f \rangle$ is equivalent to

$$\langle x^*, f' - \mu f \rangle = (\lambda - \mu) \langle x^*, f \rangle ,$$

which is equivalent, by taking $g = e^{-\mu \bullet} f$, to

$$\langle e^{\mu \bullet} x^*, g' \rangle = (\lambda - \mu) \langle e^{\mu \bullet} x^*, g \rangle . \tag{1}$$

On the other hand, taking $u^*(a) = \int_{-r}^{a} e^{\mu s} x^*(s) ds$, one obtains by integration by parts

$$\langle e^{\mu \bullet} x^*, g \rangle = \langle u^*(0), g(0) \rangle - \langle u^*, g' \rangle . \tag{2}$$

Writing $g(a) = g(0) - \int_a^0 g'(s) ds$, we deduce from $g(a) = e^{-\mu a} f(a)$:

$$f(a) = e^{\mu a} g(0) - e^{\mu a} \int_a^0 g'(s) ds,$$

hence, since $g(0) = f(0) = \Phi f$,

$$g(0) = \Phi_\mu(g(0)) - \Phi \left(e^{\mu \bullet} \int_\bullet^0 g'(s) ds \right) .$$

So, for μ sufficiently large μ in $\rho(A_\Phi)$, we have

$$\begin{aligned} g(0) &= -(1 - \Phi_\mu)^{-1} \Phi \left(e^{\mu \bullet} \int_\bullet^0 g'(s) ds \right) \\ &= -[(1 - \Phi_\mu)^{-1} \, \bar{\Phi}_\mu \, \mathcal{I}] g', \end{aligned} \tag{3}$$

where $\mathcal{I} h := \int_\bullet^0 h(s) ds$ and $\bar{\Phi}_\mu \, (h) := \Phi \left(e^{\mu \bullet} h \right)$ for $h \in X$.

Replacing the value of $g(0)$ in formula (2) by the value obtained in (3) and putting the result in (1) gives, for every $g' \in X$,

$$\begin{aligned} \langle e^{\mu \bullet} x^*, g' \rangle &= -(\lambda - \mu) \left(\left\langle u^*(0), [(1 - \Phi_\mu)^{-1} \, \bar{\Phi}_\mu \, \mathcal{I}] g' \right\rangle + \langle u^*, g' \rangle) \right) \\ &= -(\lambda - \mu) \left\langle [(1 - \Phi_\mu)^{-1} \, \bar{\Phi}_\mu \, \mathcal{I}]^* u^*(0) - u^*, g' \right\rangle . \end{aligned}$$

So, denoting $\bar{\Psi}_\mu = (1 - \Phi_\mu)^{-1} \, \bar{\Phi}_\mu$, one has

$$x^*(a) = -(\lambda - \mu) e^{-\mu a} [(\bar{\Psi}_\mu \, \mathcal{I})^* u^*(0)](a) - (\lambda - \mu) e^{-\mu a} u^*(a).$$

To prove the differentiability of x^* we have to investigate the smoothness of $[(\bar{\Psi}_\mu \, \mathcal{I})^* u^*(0)](a)$. Since

$$\begin{aligned} \left\langle (\bar{\Psi}_\mu \, \mathcal{I})^* u^*(0), h \right\rangle &= \left\langle \bar{\Psi}_\mu^* \, u^*(0), \mathcal{I} h \right\rangle \\ &= \int_{-r}^{0} \left\langle [\bar{\Psi}_\mu^* \, u^*(0)](a), \int_a^0 h(s) ds \right\rangle da \\ &= \int_{-r}^{0} \left\langle \int_{-r}^{a} [\bar{\Psi}_\mu^* \, u^*(0)](s) ds, h(a) \right\rangle da \\ &= \left\langle \int_{-r}^{\bullet} [\bar{\Psi}_\mu^* \, u^*(0)](s) ds, h \right\rangle , \end{aligned}$$

we obtain, $[(\bar{\Psi}_\mu\ \mathcal{I})^* u^*(0)](a) = \int_{-r}^{a} [\bar{\Psi}_\mu^*\ u^*(0)](s)ds$, which shows that x^* is differentiable with derivative in X^*.

2) For $f \in D(A_\Phi)$, we have $\langle A_\Phi^* x^*, f\rangle = \langle x^*, f'\rangle$. We can now integrate by parts and obtain

$$\begin{aligned}\langle A_\Phi^* x^*, f\rangle &= \langle x^*(0), f(0)\rangle - \langle x^*(-r), f(-r)\rangle - \langle x^{*\prime}, f\rangle \\ &= \langle -x^{*\prime} + \Phi^* x^*(0), f\rangle - \langle x^*(-r), f(-r)\rangle .\end{aligned}$$

Using $A_\Phi^* x^* = \lambda x^*$, one obtains

$$\langle \lambda x^*, f\rangle = \langle -x^{*\prime} + \Phi^* x^*(0), f\rangle - \langle x^*(-r), f(-r)\rangle , \tag{4}$$

for any $f \in D(A_\Phi)$. Since $D(A_\Phi)$ is dense in X, (4) is equivalent to

$$\begin{cases} x^{*\prime} + \lambda x^* = \Phi^* x^*(0) \\ x^*(-r) = 0 \end{cases}$$

which gives

$$x^*(a) = e^{-\lambda a} \int_{-r}^{a} e^{\lambda s} [\Phi^* x^*(0)](s) ds,$$

and, for $a = 0$, $x^*(0) = \int_{-r}^{0} e^{\lambda s} [\Phi^* x^*(0)](s)ds$. So, for any $z \in Y$,

$$\begin{aligned}\langle x^*(0), z\rangle &= \int_{-r}^{0} \left\langle e^{\lambda s}[\Phi^* x^*(0)](s), z\right\rangle ds \\ &= \int_{-r}^{0} \left\langle [\Phi^* x^*(0)](s), e^{\lambda s} z\right\rangle ds \\ &= \left\langle \Phi^* x^*(0), e^{\lambda \bullet} z\right\rangle \\ &= \langle x^*(0), \Phi_\lambda z\rangle \\ &= \langle \Phi_\lambda^* x^*(0), z\rangle ,\end{aligned}$$

hence $x^*(0) = \Phi_\lambda^*[x^*(0)]$. ■

Proposition 4.3. *Let y be any element of Y. If, for $\lambda \in \rho(A_\Phi)$ large enough, $(1 - \Phi_\lambda)$ is invertible, then*

i) $l_\lambda^{-1} y = \varepsilon_\lambda \otimes (1 - \Phi_\lambda)^{-1} y$,

ii) $\lambda_y = \lambda_0 - \langle x_0^*(0), y\rangle$ with $x_0^*(0)$ satisfying $x_0^*(0) = \Phi_{\lambda_0}^*[x_0^*(0)]$.

Proof. i) For $\lambda \in \rho(A_\Phi)$, $f = l_\lambda^{-1} y$ means that $y = lf$ and $f \in \ker(\lambda - A)$, that is $y = f(0) - \Phi f$ and $f = \varepsilon_\lambda \otimes f(0)$. So,

$$y = f(0) - \Phi(\varepsilon_\lambda \otimes f(0)) = (1 - \Phi_\lambda) f(0).$$

Hence, if for some $\lambda \in \rho(A_\Phi)$, $(1 - \Phi_\lambda)$ is invertible, then $f(0) = (1 - \Phi_\lambda)^{-1} y$ and

$$l_\lambda^{-1} y = f = \varepsilon_\lambda \otimes (1 - \Phi_\lambda)^{-1} y.$$

ii) We have $\lambda_y = \lambda_0 - (\lambda - \lambda_0)\left\langle x_0^*, l_\lambda^{-1} y\right\rangle$. From i) and Proposition 4.2, we know that

$$\begin{aligned}\left\langle x_0^*, l_\lambda^{-1} y\right\rangle &= \int_{-r}^{0} \left\langle e^{-\lambda_0 a} \int_{-r}^{a} e^{\lambda_0 s} [\Phi^* x_0^*(0)](s) ds, e^{\lambda a} (1 - \Phi_\lambda)^{-1} y\right\rangle da \\ &= \left\langle \int_{-r}^{0} \left(e^{(\lambda - \lambda_0) a} \int_{-r}^{a} e^{\lambda_0 s} [\Phi^* x_0^*(0)](s) ds\right) da, (1 - \Phi_\lambda)^{-1} y\right\rangle \\ &= \frac{1}{\lambda - \lambda_0} \left\langle \int_{-r}^{0} e^{\lambda_0 s} [\Phi^* x_0^*(0)](s) ds - \int_{-r}^{0} e^{\lambda s} [\Phi^* x_0^*(0)](s) ds, (1 - \Phi_\lambda)^{-1} y\right\rangle\end{aligned}$$

However, we have

$$x_0^*(0) = \int_{-r}^{0} e^{\lambda_0 s}[\Phi^* x_0^*(0)](s)ds \text{ and } \Phi_\lambda^*[x_0^*(0)] = \int_{-r}^{0} e^{\lambda s}[\Phi^* x_0^*(0)](s)ds,$$

hence,

$$\begin{aligned}\left\langle x_0^*, l_\lambda^{-1} y\right\rangle &= \frac{1}{\lambda-\lambda_0}\left\langle x_0^*(0) - \Phi_\lambda^*[x_0^*(0)], (1-\Phi_\lambda)^{-1} y\right\rangle \\ &= \frac{1}{\lambda-\lambda_0}\left\langle x_0^*(0), y\right\rangle\end{aligned}$$

and $\lambda_y = \lambda_0 - \langle x_0^*(0), y\rangle$. ■

Definition 4.4. Let Φ be a bounded linear operator from X into Y and $\Phi_\lambda : Y \to Y$ be the operator defined by $\Phi_\lambda(z) := \Phi(e^{\lambda\bullet} z)$. We will say that Φ is of compact type if Φ_λ has a compact iterate for all $\lambda \in \mathbb{C}$.

We Assume in what follows that the dual space of X is $X^* = L^\infty(]-r,0[\,,Y^*)$ if $p=1$ and $X^* = L^q(]-r,0[\,,Y^*)$ where $\frac{1}{p}+\frac{1}{q}=1$ if $p>1$.

Proposition 4.5. *[1] Let Y be a Banach lattice and $\Phi : X \to Y$ a positive bounded linear operator. Assume that Φ is compact or that Φ is of compact type and $\{(T_\Phi(t)f)(s);\ f \in B\}$ is relatively compact for $s \in\,]-r,0[$ (B is the unit ball of X). Assume moreover that Φ is a uniform limit of finite rank operators and that Φ_λ is irreducible for some $\lambda \in \mathbb{R}$. Then, the equation $r(\Phi_\lambda) = 1$ has a unique solution $\lambda_0 = s(A_\Phi) = \omega_0(A_\Phi)$, there is a unique (when normalized) strictly positive eigenvector x_0^* of A_Φ^* associated to λ_0, and there exists a positive projection of rank one P such that*

$$\left\| e^{-\lambda_0 t} T_\Phi(t) - P \right\| \le M e^{-\delta t}$$

for suitable constants $\delta > 0$, $M \ge 1$ and all $t \ge 0$.

Theorem 4.6. *Under the same hypotheses as in Proposition 4.5, we have*

$$\left\| e^{-\lambda_0 t} T_\Phi(t) - P \right\| \le M e^{-\alpha t}$$

for $M \ge 1$ and any positive $\alpha < \alpha_0 = \sup\{\langle x_0^*(0), y\rangle;\ y \in Y_+,\ \Phi - x_0^* \otimes y \ge 0\}$, where $x_0^*(0)$ is a strictly positive eigenvector of Φ_λ^* associated to the eigenvalue 1, and x_0^* given by $x_0^*(a) = e^{-\lambda_0 a}\int_{-r}^{a} e^{\lambda_0 s}[\Phi^* x_0^*(0)](s)ds$ is an eigenvector of A_Φ^* associated to λ_0.

Proof. Writing $R(\lambda, A_y)$ in the same way as in Proposition 3.8, we see that the semigroup $(T_y(t))_{t\ge 0}$ is positive if and only if the operator $(\Phi - x_0^* \otimes y)$ is positive. On the other hand, by assumption, the semigroup $(T_\Phi(t))_{t\ge 0}$ is eventually compact (see [1]), hence (Proposition 3.4 ii)) $(T_y(t))_{t\ge 0}$ is also eventually compact. So, $\omega_e(T_y) < \omega_0(T_y)$. and then (see [7]), we have $\omega_0(T_y) = s(A_y)$, $\sigma_0(A_y) = \{s(A_y)\}$ and there exists $v \in X_+$, $v \ne 0$ such that $A_y v = s(A_y)v$. By the fact that x_0^* is strictly positive one has $\langle x_0^*, v\rangle \ne 0$

Now, using Propositions 3.7 and 4.3 ii), we obtain that $\omega_0(T_y) = s(A_y) = \lambda_y = \lambda_0 - \langle x_0^*(0), y\rangle$, for any $y \in Y$ such that $\Phi - x_0^* \otimes y \ge 0$. If, moreover $y \in Y_+$, $y \ne 0$, then $\langle x_0^*(0), y\rangle > 0$ and so, $s(A_y) < \lambda_0$.

Finally, since $\sigma(A_y) = (\sigma(A_\Phi)\setminus\{\lambda_0\})\cup\{\lambda_y\}$ (Proposition3.6) and $s(A_y) = \lambda_y$, we have $s(A_y) \geq s(\widetilde{A}_\Phi)$ where $\widetilde{A}_\Phi$ is the restriction of A_Φ to $S = R(\lambda_0 - A_\Phi)$. On the other hand we have $s(A_y) = \omega_0(T_y) \geq \omega_e(\widetilde{T}_\Phi)$. Hence $s(A_y) = \omega_0(\widetilde{T}_\Phi)$ and any number α such that $0 < \alpha < \langle x_0^*(0), y\rangle = \lambda_0 - s(A_y)$ is an estimate of the rate of convergence of $e^{-\lambda_0 t}T_\Phi(t)$ to P. ■

5. Application

In this section we will apply the results of Section 4 to an equation in demography.

Let us consider a population, divided into n states. The states correspond for example to a classification of individuals by geographical habitat, social status or other criteria (see [16], [10] for more details).

We will denote by $u(a,t) := (u_i(a,t))_{1\leq i\leq n}$ the vector population densities with $\int_{a_1}^{a_2} u_i(a,t)dt$ the total subpopulation in the state i, $1 \leq i \leq n$, with age between a_1 and a_2 at time t,

$Q(a) := (q_{ij}(a))$, $1 \leq i,j \leq n$, the matrix of transition between states. For $i \neq j$, $q_{ij}(a) \geq 0$ is the instantaneous rate of transition, at age a, from the state j to the state i,

$q_{ii}(a) := -\mu_i(a) - \sum_{j\neq i} q_{ji}(a)$, $\mu_i(a) \geq 0$ the death rate at age a of individuals in state i, $1 \leq i \leq n$,

$M(a) := (m_{ij}(a))$ $1 \leq i,j \leq n$, the fertility matrix, where $m_{ij}(a) \geq 0$ is the average number of offsprings per unit of time in state i produced by an individual at age a in state j.

The dynamics of such a population is described by the following vector type Lotka–von Foerster system of equations:

$$(L.F.E.) \quad \begin{cases} \frac{\partial}{\partial t}u(a,t) + \frac{\partial}{\partial a}u(a,t) = Q(a)u(a,t), \\ u(0,t) = \int_0^{a_r} M(a)u(a,t)da, \\ u(a,0) = p(a), \end{cases}$$

where a_r is the maximum reproductive age ($M(a) \equiv 0$ for all $a > a_r$) and $p(a) = (p_i(a))_{1\leq i\leq n}$ is the initial population vector.

Let a_s be the maximum life span of the population ($a_r < a_s \leq +\infty$). The natural state space for the density function $u(a,t)$ is the space $X = L^1([0,r], \mathbb{R}^n)$, where r is a positive real number, such that $a_r \leq r < a_s$. With the condition $r < a_s$, we can have $\int_0^{a_s} \mu_i(a)da = +\infty$, which means that the survival function $\exp(-\int_0^a \mu_i(x)dx)$ is equal to zero for $a = a_s$.

We assume that the functions $m_{ij}(a) \geq 0$, for $i,j = 1,\cdots,n$, are in $L^\infty(]0,r[, \mathbb{R}^n)$ ($m_{ij}(a) = 0$ for $a \geq a_r$). The functions $\mu_i(a)$ and $q_{ij}(a)$ are assumed to be positive and continuous on $]0,r[$.

To have a representation of the solution of $(L.F.E.)$ as a translation semigroup, generated by an operator A_Φ with boundary conditions, we introduce the matrix $\Pi(a)$, defined as a solution of the following matrix differential equation:

$$\begin{cases} \frac{d}{da}\Pi(a) = Q(a)\Pi(a), \\ \Pi(0) = I. \end{cases}$$

Then $\Pi(a)$ is the survival matrix. The entry $\pi_{ij}(a)$ of $\Pi(a)$ represents the rate that an individual born in the state j survives and is in the state i at age a.

By Liouville's formula, we have

$$\det \Pi(a) = \exp\left(\int_0^a \sum_{i=1}^n q_{ii}(a)\right) \neq 0.$$

So, $\Pi(a)$ is invertible. Its inverse $\Pi^{-1}(a)$ satisfies the equation:

$$\begin{cases} \frac{d}{da}\Pi^{-1}(a) = -\Pi^{-1}(a)Q(a), \\ \Pi^{-1}(0) = I. \end{cases}$$

Now, taking $f := \Pi^{-1}(a)u(a, \bullet)$, $(L.F.E.)$ can be formulated as the following abstract Cauchy problem in $X = L^1([0,r], I\!R^n)$:

$$\begin{cases} \frac{d}{dt}f = A_\Phi f, \\ f(0) = p, \end{cases}$$

where A_Φ is the population operator defined by

$$A_\Phi f := f', \quad D(A_\Phi) := \left\{f \in W^{1,1}([0,r], I\!R^n);\ f(0) = \Phi(f)\right\}, \tag{5}$$
$$\Phi : X \to I\!R^n, \quad \Phi f := \int_0^r B(a)f(a)da, \quad B(a) := M(a)\Pi(a).$$

Here, $B(a)$ is the net fecundity matrix. It is not difficult to check that $B(a)$ is a nonnegative matrix.

The semigroup formulation represents the definition of the evolution in time of the initial population vector. The limit condition in the $A.E.G.$ reflects the phenomenon that the age distribution approaches a stable shape independent of the initial population. The number λ_0 is an intrinsic constant characterizing the studied species in their environment.

In the following proposition we summarize some known (see [16] for instance) properties of the population operator A_Φ and the associated semigroup $(T_\Phi(t))_{t\geq 0}$.

Proposition 5.1. *Let A_Φ be the population operator defined by (5) and let $m := ess \sup\{\|M(a)\|;\ a \in]0,r[\}$, $d = \inf\{\mu_i(a);\ 1 \leq i \leq n,\ a \in]0,r[\}$. Then the following holds.*

i) The spectrum of A_Φ is point spectrum and $\sigma(A_\Phi) = \{\lambda \in \mathbb{C};\ \det(I - \Phi_\lambda) = 0\}$, where Φ_λ is the $n \times n$ matrix given by $\Phi_\lambda = \int_0^r e^{-\lambda a}B(a)da$.

ii) Any $\lambda \in \sigma(A_\Phi)$ is a pole of $R(\lambda, A_\Phi)$ with finite algebraic multiplicity. The geometric eigenspace of A_Φ ,iven by $N(\lambda - A_\Phi) = \{f;\ f(a) = e^{\lambda a}x,\ x \in N(I - \Phi_\lambda)\}$.

iii) For any $\lambda > d-m$, w ıave $\lambda \in \rho(A_\Phi)$ and $R(\lambda, A_\Phi)$ is a positive compact operator.

iv) A_Φ is the infinitesimal generator of a positive semigroup $(T_\Phi(t))_{t\geq 0}$ satisfying $\|T_\Phi(t)\| \leq e^{(m-d)t}$ and $T_\Phi(t)$ compact for $t \geq r$.

We now apply Theorem 4.6 to this situation

Theorem 5.2. *Assume that for some* $\lambda \in I\!R$, *the matrix* $\Phi_\lambda = \int_0^r e^{-\lambda a} B(a)da$ *is irreducible. Then, the semigroup* $(T_\Phi(t))_{t\geq 0}$ *has A.E.G.. Its intrinsic growth rate* λ_0 *is the unique real solution of* $r(\Phi_\lambda) = 1$. *If we define*

$$\alpha_0 := \sup\left\{\left\langle v^T, y\right\rangle : y \in I\!R^n_+,\ e^{\lambda_0 a}B(a) - \left[yv^T\right]\int_0^a e^{\lambda_0 a}B(a)da \geq 0\right\},$$

then for every α, $0 < \alpha < \alpha_0$ *there exists* M *such that*

$$\left\| e^{-\lambda_0 t}T_\Phi(t) - P\right\| \leq Me^{-\alpha t}$$

where v^T *is a strictly positive eigenvector of* $\Phi^T_{\lambda_0}$ *associated to the eigenvalue* 1.

Proof. In our situation the operator Φ is defined from X to $I\!R^n$ by $\Phi f := \int_0^r B(a)f(a)da$, $B(a) := M(a)\Pi(a)$ and Φ_λ is the $n \times n$ matrix given by $\Phi_\lambda := \int_0^r e^{-\lambda a}B(a)da$. So, it is easy to see that the hypotheses of Theorem 4.6 are satisfied. Let us now write the condition $\Phi - x_0^* \otimes y \geq 0$ using the data of our example. We have (see Theorem 4.6): $x_0^*(a) = e^{-\lambda_0 a}\int_0^a e^{\lambda_0 s}[\Phi^* x_0^*(0)](s)ds$, $x_0^*(0) = \Phi^T_{\lambda_0}(x_0^*(0))$ and for any row vector z^T, $[\Phi^* z^T](s) = z^T B(s)$. Hence, setting $v^T := x_0^*(0)$, we obtain $x_0^*(a) = e^{-\lambda_0 a}\int_0^a e^{\lambda_0 s}v^T B(s)ds = e^{-\lambda_0 a}v^T\int_0^a e^{\lambda_0 s}B(s)ds$. So, for any $f \in X$,

$$\begin{aligned}(\Phi - x_0^* \otimes y)f &= \int_0^r B(a)f(a)da - \left[\int_0^r (e^{-\lambda_0 a}v^T\int_0^a e^{\lambda_0 s}B(s)ds)f(a)da\right]y\\ &= \int_0^r \left\{B(a) - e^{-\lambda_0 a}[yv^T]\int_0^a e^{\lambda_0 s}B(s)ds\right\}f(a)da\\ &= \int_0^r \left\{e^{\lambda_0 a}B(a) - [yv^T]\int_0^a e^{\lambda_0 s}B(s)ds\right\}e^{-\lambda_0 a}f(a)da.\end{aligned}$$

Consequently, $\Phi - x_0^* \otimes y \geq 0$ is equivalent to $e^{\lambda_0 a}B(a) - [yv^T]\int_0^a e^{\lambda_0 s}B(s)ds \geq 0$ for a.e. $a \in]0, r[$. ∎

Remark. The matrix $[yv^T]$ is of the form

$$[yv^T] = \begin{bmatrix} y_1v_1 & y_1v_2 & \cdots & y_1v_n\\ y_2v_1 & y_2v_2 & & y_2v_n\\ \vdots & & & \\ y_nv_1 & y_nv_2 & & y_nv_n\end{bmatrix}$$

with $v_i > 0$, $i = 1, \ldots, n$. The vector y must be chosen such that the matrix $e^{\lambda_0 a}B(a) - [yv^T]\int_0^a e^{\lambda_0 s}B(s)ds$ becomes nonnegative. So, if the entry $b_{ij}(a) \neq 0$ of $B(a)$ is equal to zero a.e. on some interval $I \subset [\varepsilon, r]$, $0 < \varepsilon < r$, then y_i must be equal to zero.

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