Journal of Differential Equations **181**, 1–30 (2002)
doi:10.1006/jdeq.2001.4076, available online at http://www.idealibrary.com on IDEAL®

Semigroup Properties and the Crandall Liggett Approximation for a Class of Differential Equations with State-Dependent Delays

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Received May 27, 1999; revised May 1, 2001

We present an approach for the resolution of a class of differential equations with state-dependent delays by the theory of strongly continuous nonlinear semigroups. We show that this class determines a strongly continuous semigroup in a closed subset of $C^{0,1}$. We characterize the infinitesimal generator of this semigroup through its domain. Finally, an approximation of the Crandall–Liggett type for the semigroup is obtained in a dense subset of $(C, \|\cdot\|_\infty)$. As far as we know this approach is new in the context of state-dependent delay equations while it is classical in the case of constant delay differential equations.

1. INTRODUCTION

We consider the differential equation with state-dependent delays,

$$(1) \qquad \begin{cases} x'(t) = f(x(t - r(x_t))), & \text{for} \quad t \geqslant 0 \\ x_0 = \varphi \in C^{0,1}, \end{cases}$$

where f is a function from IR into IR and r is a function from C into $[0, M]$ (here $C := C([-M, 0], IR)$ is the Banach space of continuous functions from $[-M, 0]$ into IR, endowed with the norm $\|\cdot\|_\infty$, M is a positive constant). Finally, we denote by x_t the element of C defined by

$$x_t(\theta) = x(t + \theta), \qquad \text{for} \quad \theta \in [-M, 0].$$

0022-0396/02 $35.00

The notation $C^{0,1} = C^{0,1}([-M, 0]; IR)$ stands for the Banach space of Lipschitz continuous functions from $[-M, 0]$ into IR, endowed with the norm

$$\|\varphi\|_{0,1} = \max\{\|\varphi\|_\infty, \|\varphi'\|_{L^\infty}\}.$$

Differential equations with state-dependent delays arise in various applications, in particular, in mathematical ecology and bio-economics, see notably the work of Bélair [4], Arino *et al.* [2], and Aiello *et al.* [1].

Qualitative and quantitative studies of these equations developed actively within the last ten years. At the qualitative level, Mallet-Paret *et al.* [8], Kuang and Smith [7], and Arino *et al.* [3] discuss existence of periodic and slowly oscillating periodic solutions. As for the quantitative aspects, state-dependent delay equations brings specific problems, the Cauchy problem associated with these equations is not well posed in the space of continuous functions, due to the non-uniqueness of solutions whatever the regularity of the functions f and r. Uniqueness holds in $C^{0,1}$; however, the equation does not yield a strongly continuous semigroup in this space either (see Section 3, Proposition 5(b)).

In this work, we present an approach by the theory of strongly continuous nonlinear semigroups. We show that Eq. (1) determines a strongly continuous semigroup in a closed subset of $C^{0,1}$. We characterize the infinitesimal generator of this semigroup in terms of its domain. Finally, an approximation of the Crandall–Liggett type for the semigroup is obtained in a dense subset of $(C, \|\cdot\|_\infty)$. As far as we know this approach is new in the context of state-dependent delay equations while it is classical in the case of constant delay differential equations; see Webb [14], and Dyson and Villella-Bressan [6]. For the case of neutral delay differential equations, we refer the reader to Arino and Sidki [12], and Plant [11].

The paper is organized in six sections including the introduction. In the second section we recall some basic framework related to semigroup theory. Section 3 deals with the nonlinear semigroup solution of Eq. (1). Section 4 investigates smoothness properties of the equation. Section 5 is devoted to the characterization of the infinitesimal generator of the semigroup. In the last section we present the result about the Crandall–Liggett approximation of this semigroup.

2. PRELIMINARIES

Equation (1) can be written in the following form,

$$\begin{cases} x' = F(x_t) \\ x_0 = \varphi, \end{cases} \tag{2}$$

where

$$F(\varphi) = f(\varphi(-r(\varphi))), \qquad \text{for} \quad \varphi \in C. \tag{3}$$

Notice that the functional F is defined on C, but it is clear that it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of f and r.

Throughout the paper f and r satisfy part or all of the assumptions:

(H_1) $f\colon IR \to IR$ is locally Lipschitz continuous and $r\colon C \to [0, M]$ is Lipschitz continuous on the bounded subsets of C.

(H_2) There exist two constants α and β such that $|f(x)| \leqslant \alpha\,|x| + \beta$, for every $x \in IR$.

(H_3) The functions f and r are of class C^1.

We denote $lip(h)$ the Lipschitz constant of any Lipschitz continuous function h.

For each $k > 0$ we denote:

$$lip_k(f) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|};\, 0 \leqslant |x|, |y| \leqslant k\right\},$$

$$lip_k(r) = \sup\left\{\frac{|r(\varphi) - r(\psi)|}{\|\varphi - \psi\|_\infty};\, 0 \leqslant \|\varphi\|_\infty, \|\psi\|_\infty \leqslant k\right\},$$

Finally, we denote C^1, respectively C^2, the space of continuously differentiable functions on $[-M, 0]$, (resp. twice continuously differentiable functions), endowed with the natural norm derived from the sup norm.

By a solution of Eq. (2), we mean a function x defined on $[-M, a]$ for some a positive such that x is continuous on its domain, differentiable on $]0, a]$ and satisfies Eq. (2).

THEOREM 2.1 [8]. *Suppose* H_1 *and* H_2 *hold. Then for each initial datum* $\varphi_0 \in C^{0,1}$, *Eq.* (2) *has a unique solution* $x^{\varphi_0}(t)$ *defined on* $[-M, \infty[$.

Remark 2.2. (a) If H_1 is satisfied, then for every $k > 0$ and every $\varphi_0 \in C^{0,1}$ and $\varphi_1 \in C$ such that $0 \leqslant \|\varphi_0\|_\infty, \|\varphi_1\|_\infty \leqslant k$ we have

$$|F(\varphi_1) - F(\varphi_0)| \leqslant lip_k(f)\{lip_k(r)\, lip(\varphi_0) + 1\}\, \|\varphi_1 - \varphi_0\|_\infty. \tag{4}$$

(b) Assumption H_3 is useful in showing that the restriction of F to the space of C^1-functions is continuously differentiable.

We will now recall some definitions and results related to the Crandall–Liggett approximation.

DEFINITION 2.3 [13]. Let $S(t), t \geqslant 0$ be a family of operators in a Banach space X. We say that $S(t), t \geqslant 0$, defines a strongly continuous semigroup in a closed subset Y of X if

(i) $S(t)$ is continuous from Y into Y, for each $t \geqslant 0$,

(ii) $S(0) = I_Y$, $S(t+s) = S(t) \circ S(s)$, for each $t, s \geqslant 0$, $x \in Y$

(iii) $t \longmapsto S(t)\, x$ is continuous from $[0, \infty[$ into Y, for each fixed $x \in Y$.

DEFINITION 2.4 [13]. The infinitesimal generator of the semigroup $S(t), t \geqslant 0$, is the operator Λ: $Y \to X$ given by

$$\Lambda x = \lim_{t \to 0} \frac{1}{t}\, (S(t)\, x - x),$$

defined at each $x \in Y$ where this limit exists.

Denote $D(\Lambda)$ the domain of Λ.

It is well known, see for instance [10, 13], that in the linear case the limit exists at each point of a dense subset of X. However, in the nonlinear case, the generator does not exist necessarily and the domain may be empty [5].

DEFINITION 2.5 [10, 13]. Let X be a Banach space and Λ an operator defined on a subset of X with values into X. We say that Λ

(a) is accretive if $\|(I + \lambda\Lambda)\, x - (I + \lambda\Lambda)\, y\| \geqslant \|x - y\|$, for each $x, y \in D(\Lambda)$, $\lambda > 0$,

(b) is μ-accretive if $\Lambda + \mu I$ is accretive, for some $\mu > 0$.

THEOREM 2.6 [5]. *Let Λ be an operator with domain contained in X. If there exists $\mu \in IR$, such that Λ is μ-accretive and the range of $I + \lambda\Lambda$, denoted $R(I + \lambda\Lambda)$, is equal to X for each $\lambda > 0$ small enough, then*

$$\lim_{n \to 0} (I + (t/n)\, \Lambda)^{-n}\, x := S(t)\, x$$

exists, for $x \in \overline{D(\Lambda)}$, $t \geqslant 0$. Furthermore, $S(t), t \geqslant 0$ is a strongly continuous semigroup on $\overline{D(\Lambda)}$ such that

$$\|S(t)\, x - S(t)\, y\| \leqslant \exp(\mu t)\, \|x - y\| \qquad \textit{for every} \quad x, y \in \overline{D(\Lambda)}, t \geqslant 0.$$

3. THE SEMIGROUP ASSOCIATED WITH EQ. (2)

Consider the family of operators $T(t), t \geqslant 0$ defined by $T(t)\, \varphi = x_t^\varphi$, for each $\varphi \in C^{0,1}$, where $x^\varphi(t)$ is the solution of Eq. (2). In the sequel, we will show that $T(t)$ is a strongly continuous semigroup in a closed subset $\mathbf{E}$ of $C^{0,1}$.

We first show that strong continuity does not occur in the space $C^{0,1}$.

PROPOSITION 3.1. *Suppose* H_1 *and* H_2 *hold. Then,*

(a) *the family of operators* $T(t)$, $t \geqslant 0$ *satisfies*

$$T(0) = I,$$

and

$$T(t+s) = T(t)\,T(s), \qquad \forall t, s \geqslant 0,$$

$$\textit{that is,} \qquad (T(t))_{t \geqslant 0} \textit{ is analgebraic semigroup.}$$

(b) $(T(t))_{t \geqslant 0}$ *is not strongly continuous in* $C^{0,1}$

Proof. (a) This assertion follows from the existence and uniqueness of the solution of Eq. (2).

(b) Choose as an initial function

$$\varphi_0(\theta) = \begin{cases} \theta + M & \text{if } \theta \in \left[-M, -\dfrac{M}{2}\right] \\ \dfrac{M}{2} & \text{if } \theta \in \left[-\dfrac{M}{2}, 0\right]. \end{cases}$$

We select a number t_0, $0 < t_0 < \frac{M}{2}$ and a second number t, $0 < t \leqslant t_0$, that will further be moved towards t_0. Put $x_t = x_t^{\varphi}$. We have

$$x_t(\theta) = \begin{cases} \dfrac{M}{2} & \text{if } \theta \in \left[-\dfrac{M}{2} - t, -t\right] \\ \theta + M + t & \text{if } \theta \in \left[-M, -\dfrac{M}{2} - t\right]. \end{cases}$$

So, we have

$$(x_{t_0} - x_t)(\theta) = \begin{cases} t_0 - t & \text{if } \theta \in \left[-M, -\dfrac{M}{2} - t_0\right] \\ -\theta - \dfrac{-M}{2} - t & \text{if } \theta \in \left[-\dfrac{M}{2} - t_0, -\dfrac{M}{2} - t\right] \\ 0 & \text{if } \theta \in \left[-\dfrac{M}{2} - t, -t_0\right]. \end{cases}$$

Observe that $\|\frac{d}{d\theta}(x_{t_0} - x_t)\|_{L^\infty} \geqslant 1$. Then we have

$$\lim_{t \to t_0} \|T(t_0)\,\varphi_0 - T(t)\,\varphi_0\|_{0,1} \geqslant 1.$$

We conclude that

$$\begin{aligned} T(\cdot)\,\varphi_0 \colon [0, \infty[&\to (C^{0,1}, \|\cdot\|_{0,1}) \\ t &\longmapsto T(t)\,\varphi_0, \end{aligned}$$

is not continuous. More precisely, we have proved that $T(\cdot)\,\varphi_0$ is not continuous at any point $t_0 \in]0, \frac{M}{2}[$. ▮

Note that the example built in b) of proposition 3.1 is independent of the equation.

We now consider the subset $\mathbf{E}$ of $C^{0,1}$ defined by

$$\text{(5)} \quad \mathbf{E} = \{\phi \in C^{0,1} : t \to T(t)\,\phi \text{ is continuous from } IR^+ \text{ into } (C^{0,1}, \|\cdot\|_{0,1})\}.$$

Remark 3.2. The set $\mathbf{E}$ is non-empty, $\mathbf{E}$ contains the set

$$C_F^1 = \{\varphi \in C^1 : \varphi'^-(0) = F(\varphi)\}, \tag{6}$$

where $\varphi'^-$is the left hand derivative. Under the assumption (H_1), C_F^1 is a closed subset of C^1, dense in the space C. Moreover, C_F^1 is a locally Lipschitz submanifold of C^1, a property that will not used in this work.

PROPOSITION 3.3.

$$\mathbf{E} = C_F^1.$$

which in particular entails that $\mathbf{E}$ *is a subset of* C^1.

Proof. In view of Remark 3.2, it is sufficient to show that

$$C_F^1 \supset \mathbf{E}.$$

Let $\varphi \in \mathbf{E}$, and $t_0 \geqslant 0$. Put $x(t) = x^\varphi(t)$. We start by showing that x'_{t_0} is equal almost everywhere (with respect to the Lebesgue measure λ) to a continuous function in $[-M, 0]$. This will be done in two steps: in step 1, we show that x'_{t_0} is continuous on $[-M+\varepsilon, 0]$ for any $\varepsilon > 0$ small enough, and in step 2, we show the continuity of x'_{t_0} on $[-M, -\varepsilon]$ for any $\varepsilon > 0$ small enough. Continuity on $[-M, 0]$ follows directly.

Step 1. Note that, φ being in $\mathbf{E}$ implies that $t \to T(t)\,\varphi = x_t$ is continuous from IR^+ into $C^{0,1}$, which yields two consquences:

(i) continuity of $t \to x_t$, from IR^+ into C;

(ii) continuity of $t \to x'_t$, from IR^+ into L^∞.

Given $\varepsilon > 0$, $n_0 > \frac{1}{\varepsilon}$. We define the sequence of functions $(g_n)_{n \geqslant n_0}$ on $[-M+\varepsilon, 0]$ by

$$g_n(\theta) = n \int_0^{\frac{1}{n}} x'(t_0 + \theta - u)\, d\lambda(u). \tag{7}$$

The family of functions $\mathscr{F} = \{g_n : n \geqslant n_0\}$ is uniformly equicontinuous. In fact, for each real h small enough and $\theta \in [-M+\varepsilon, 0]$ such that $\theta + h \in [-M+\varepsilon, 0]$ and $t_0 + h \geqslant 0$ we have

$$|g_n(\theta) - g_n(\theta + h)| \leqslant \|x'_{t_0+h} - x'_{t_0}\|_{L^\infty}.$$

In view of (ii) above, we have that $\|x'_{t_0+h} - x'_{t_0}\|_{L^\infty}$ goes to zero as h goes which yields the desired equicontinuity of the sequence (g_n).

Since the functions g_n, $n \geqslant n_0$ are uniformly bounded ($\|g_n\|_\infty \leqslant \|x'_{t_0}\|_{L^\infty}$), the Ascoli theorem implies that $\mathscr{F}$ is relatively compact in C.

$$g_n(\theta) = \frac{x_{t_0}(\theta) - x_{t_0}\left(\theta - \dfrac{1}{n}\right)}{\dfrac{1}{n}}$$

being Lipschitz continuous, the function x_{t_0} is *a.e.* differentiable, therefore, for almost every $\theta \in [-M, 0]$, $g_n(\theta)$ converges towards $x'_{t_0}(\theta)$.

On the other hand, we have just shown that, for every $\varepsilon > 0$, g_n converges to some continuous function g_ε defined on $[-M+\varepsilon, 0]$, we deduce that x'_{t_0} is equal *a.e.* to a continuous function on $[-M+\varepsilon, 0]$.

Step 2. In the same manner, substituting $[-M, -\varepsilon]$ for $[-M+\varepsilon, 0]$, and

$$\overline{g_n}(\theta) = n \int_{-\frac{1}{n}}^{0} x'(t_0 + \theta - u)\, d\lambda(u),$$

for g_n, we obtain that x'_{t_0} is equal *a.e.* to a continuous function on $[-M, -\varepsilon]$. This holds for any $\varepsilon > 0$ small enough.

As a conclusion of steps 1 and 2, we thus established the existence of a continuous function g on $[-M, 0]$ such that $x'_{t_0} = g$ almost everywhere on $[-M, 0]$.

Since x_{t_0} is absolutely continuous it can be written as

$$x_{t_0}(\theta) = x_{t_0}(0) + \int_0^\theta x'_{t_0}(u)\, d\lambda(u) \tag{8}$$

$$= x_{t_0}(0) + \int_0^\theta g(u)\, du, \qquad \text{for all} \quad \theta \in [-M, 0].$$

The latter equality, with g continuous, gives that x_{t_0} is of class C^1.

In particular, $x_0 = \varphi \in C^1$. On other hand, taking $t_0 = \frac{M}{2}$, for example, and taking into account continuity of the right and left derivative of $x_{M/2}$ at $\theta = -\frac{M}{2}$, we deduce that $\varphi'^{-}(0) = F(\varphi)$. ∎

COROLLARY 3.4. *Let $\varphi \in \mathbf{E}$: $x = x^\varphi$. Then x is of class C^1 on $[-M, +\infty[$.*

PROPOSITION 3.5. *Suppose* H_1 *and* H_2 *hold. Then for each $t_0 > 0$, $k > 0$ and for each φ and ψ of $C^{0,1}$ such that $\|\varphi\|_\infty$, $\|\psi\|_\infty \leqslant k$, we have*

$$\|T(t)(\varphi) - T(t)(\psi)\|_{0,1} \leqslant \max\{\eta, 1\} \exp(\eta t)\, \|\varphi - \psi\|_{0,1}, \qquad \forall t \in [0, t_0], \tag{9}$$

where

$$\eta = \eta(\varphi, k, t_0) = lip_{\gamma_1}(f)\{lip_{\gamma_1}(r)\, \gamma_0 + 1\}, \tag{10}$$

$$\gamma_0 = \gamma_0(\varphi, k, t_0) = lip(\varphi) + \alpha(\beta t_0 + k)\, exp(\alpha t_0) + \beta, \tag{11}$$

and

$$\gamma_1 = \gamma_1(k, t_0) = (\beta t_0 + k)\, exp(\alpha t_0). \tag{12}$$

Proof. Given $t_0 > 0$, $k > 0$, let φ and ψ be two elements of $C^{0,1}$ such that $\|\varphi\|_\infty \leqslant k$, $\|\psi\|_\infty \leqslant k$. For each $t \in [0, t_0]$, if $x = x^\varphi$ is the solution of equation (2) with initial datum φ, we obtain

$$|x(t)| \leqslant |x(0)| + \int_0^t |F(x_v)|\, dv. \tag{13}$$

From assumption H_2, we have

$$|x(t)| \leqslant |x(0)| + \int_0^t (\alpha\, \|x_v\|_\infty + \beta)\, dv.$$

So,

$$|x(s)| \leqslant \|\varphi\|_\infty + \int_0^t (\alpha \|x_v\|_\infty + \beta)\, dv, \qquad 0 \leqslant s \leqslant t,$$

Then,

$$\|x_t\|_\infty \leqslant \|x_0\|_\infty + \int_0^t (\alpha \|x_v\|_\infty + \beta)\, dv.$$

By the Gronwall lemma, we obtain

$$\|x_t\|_\infty \leqslant (\|x_0\|_\infty + \beta t) \exp(\alpha t),$$

and also, for $0 \leqslant t \leqslant t_0$,

$$\|x_t^\varphi\|_\infty \leqslant (k + \beta t_0) \exp(\alpha t_0), \tag{13}$$

for each $\varphi \in C^{0,1}$ $\|\varphi\|_\infty \leqslant k$. From inequality (13), we have

$$\begin{aligned} \left| \frac{d}{dt} x^\varphi(t) \right| &= |F(x_t^\varphi)| \leqslant \alpha \|x_t^\varphi\|_\infty + \beta \\ &\leqslant \alpha(\beta t_0 + k)\, exp(\alpha t_0) + \beta, \qquad \text{for each} \quad \varphi \in C^{0,1}, \\ &\qquad \text{such that} \quad \|\varphi\|_\infty \leqslant k. \end{aligned} \tag{14}$$

Using (4), we obtain

$$|F(x_t^\varphi) - F(x_t^\psi)| \leqslant \eta \|x_t^\varphi - x_t^\psi\|_\infty,$$

where η, γ_0, and γ_1 is defined in Proposition 3.5 by (10), (11), and (12).

From the inequality

$$\|x_t^\varphi - x_t^\psi\|_\infty \leqslant \|\varphi - \psi\|_\infty + \int_0^t |F(x_t^\varphi) - F(x_t^\psi)|\, ds,$$

we deduce that

$$\|x_t^\varphi - x_t^\psi\|_\infty \leqslant \|\varphi - \psi\|_\infty + \eta \int_0^t \|x_t^\varphi - x_t^\psi\|_\infty\, ds.$$

Using again the Gronwall lemma, we obtain

$$\|x_t^\varphi - x_t^\psi\|_\infty \leqslant \exp(\eta t)\, \|\varphi - \psi\|_\infty, \qquad 0 \leqslant t \leqslant t_0.$$

We also have

$$\begin{aligned}\left|\frac{d}{dt}x^{\varphi}(t)-\frac{d}{dt}x^{\psi}(t)\right| &= |F(x_t^{\varphi})-F(x_t^{\psi})| \\ &\leqslant \eta\, \|x_t^{\varphi}-x_t^{\psi}\|_{\infty} \\ &\leqslant \eta \exp(\eta t)\, \|\varphi-\psi\|_{\infty}, \qquad 0 \leqslant t \leqslant t_0.\end{aligned}$$

By combining the above inequalities and using monotonicity of the function $\exp(\eta t)$, we deduce the desired estimations. ▮

Note that for each t, t_0, ψ, φ and η taken as in Proposition 3.5, we have

$$\|x^{\varphi}-x^{\psi}\|_{C^{0,1}([0,\, t_0])} \leqslant \max\{\eta, 1\} \exp(\eta t_0)\, \|\varphi-\psi\|_{\infty},$$

Remark 3.6. From the proof of the above proposition, we deduce that if f and r are Lipschitz continuous and if $|f|$ is bounded by a constant $\rho > 0$, we have

$$\begin{aligned}&\|T(t)(\varphi)-T(t)(\psi)\|_{0,1} \leqslant \max\{\eta, 1\}\, exp(\eta t)\, \|\varphi-\psi\|_{0,1}, \\ &\qquad t \geqslant 0, \text{ for each } \varphi, \psi \in C^{0,1},\end{aligned}$$

where $\eta = \eta(\varphi, \rho) = lip(f)\{lip(r)(lip(\varphi)+\rho)+1\}$.

COROLLARY 3.7. *Assume assumptions* H_1 *and* H_2 *be satisfied. Then,* **E** *is closed and the restriction of the family of operators* $T(t), t \geqslant 0$ *to* **E**, *also denoted* $T(t), t \geqslant 0$, *is a strongly continuous semigroup on* **E**.

For brevity, we will occasionally use the word semigroup to mean "strongly continuous semigroup".

PROPOSITION 3.8. *Suppose* H_1 *and* H_2 *hold. Then, for each* $t \in [M, \infty]$, *the operator* $T(t)$ *is completely continuous on* $(\mathbf{E}, \|\cdot\|_{0,1})$.

Proof. Given $t \geqslant M$. Continuity of $T(t)$ is taken from Proposition 3.5. Then, we only have to show that the operator is compact.

Let **B** be a bounded subset of **E**. We will show that $T(t)(\mathbf{B})$ is relatively compact. We denote $\rho = \sup\{\|\varphi\|_{0,1} \colon \varphi \in \mathbf{B}\}$.

If $\varphi \in \mathbf{B}$, and $x(t) = x^{\varphi}(t)$, is the solution of Eq. (2), then procceding in the same manner as in (13), (14), and (15), we obtain

$$\begin{cases} \|x_t\|_{\infty} \leqslant (\beta t+\rho)\, exp(\alpha t) = \gamma_1, \\ |x'(t)| \leqslant \alpha(\beta t+\rho)\, exp(\alpha t)+\beta. \end{cases}$$

Thus,

$$\|x_t'\|_\infty \leqslant \alpha(\beta t+\rho)\, exp(\alpha t)+\beta=\gamma_0. \tag{16}$$

On the other hand,

$$\begin{aligned}|x'(t+\theta_1)-x'(t+\theta_2)| &= |F(x_{t+\theta_1})-F(x_{t+\theta_2})|\\ &\leqslant lip_{\gamma_1}(f)\{lip_{\gamma_1}(r)(\gamma_0+\rho)+1\}\ \|x_{t+\theta_1}-x_{t+\theta_2}\|_\infty\\ &\leqslant lip_{\gamma_1}(f)\{lip_{\gamma_1}(r)(\gamma_0+\rho)+1\}\ \gamma_0\ |\theta_1-\theta_2|.\end{aligned}$$

We deduce that the family $\{(x_t^\varphi)' \colon \varphi\in\mathbf{B}\}$ is uniformly Lipschitz continuous with the Lipschitz constant

$$L_{\mathbf{B}} \leqslant lip_{\gamma_1}(f)\{lip_{\gamma_1}(r)(\gamma_0+\rho)+1\}\ \gamma_0. \tag{17}$$

The Ascoli Arzela theorem applied to the family $\mathscr{H}=\{(x_t^\varphi, \frac{d}{d\theta}x_t^\varphi) : \varphi\in\mathbf{B}\}$ for each $t\geqslant M$ implies that $\mathscr{H}$ is relatively compact in $C\times C$. Then $\{x_t^\varphi \colon \varphi\in\mathbf{B}\}$ is relatively compact in C^1 for each $t\geqslant M$. The conclusion follows from the fact that $\mathbf{E}$ is a closed subset of $\mathbf{C}^1$. ▮

4. SMOOTHNESS OF THE SOLUTION OF EQ. (2)

Let $\mathbf{E}_1$ be the set defined by

$$\mathbf{E}_1=\{\varphi\in C^1 : \varphi'\in C^{0,1} \text{ and } \varphi'(0)=F(\varphi)\} \tag{18}$$

PROPOSITION 4.1. *Suppose assumptions* H_1, H_2, *and* H_3 *hold. Then, for each* $\varphi_0\in\mathbf{E}_1$ *the solution* $x := x^{\varphi_0}$ *of* (2) *is* C^2 *on the interval* $[0,\infty[$. *Moreover, if we denote* x'' *the second order derivative of* x, *then we have*

$$x''(t)=f'(x(t-r(x_t)))\,x'(t-r(x_t))\{1-Dr(x_t)\,x_t'\}, \qquad t\geqslant 0. \tag{19}$$

In order to show this result, we need the following lemma:

LEMMA 4.2. *Suppose assumption* H_3 *holds. Then,* $F_{|C^1}$, *the restriction of* F *to* C^1 *is of class* C^1. *Moreover, if we denote by* L_{φ_0} *the derivative of* $F_{|C^1}$ *at* φ_0, *then*

$$\begin{aligned}L_{\varphi_0}(\psi)&=f'(\varphi_0(-r(\varphi_0)))\{\psi(-r(\varphi_0))-\varphi_0'(-r(\varphi_0))\,Dr(\varphi_0)\,\psi\},\\ &\textit{for each}\quad \psi\in C^1,\end{aligned}$$

where $Dr\colon C\to\mathscr{L}(C; IR)$ *is the Frechet derivative of the function* r.

Proof of Lemma 4.2. Let φ and φ_0 betwo elements of C^1. Using the Taylor expansion of f in the neighborhood of $\varphi_0(-r(\varphi_0))$ yields

$$F(\varphi)-F(\varphi_0)=f'(\varphi_0(-r(\varphi_0)))\{\varphi(-r(\varphi))-\varphi_0(-r(\varphi_0))\}$$
$$+o(|\varphi(-r(\varphi))-\varphi_0(-r(\varphi_0))|),$$

We then note that

$$\varphi(-r(\varphi))-\varphi_0(-r(\varphi_0))=(\varphi(-r(\varphi))-\varphi_0(-r(\varphi)))$$
$$+(\varphi_0(-r(\varphi))-\varphi_0(-r(\varphi_0)).$$

The first expression on the right can be decomposed as

$$\begin{aligned}(\varphi(-r(\varphi))-\varphi_0(-r(\varphi)))&=(\varphi-\varphi_0)(-r(\varphi_0))\\&\quad-(\varphi-\varphi_0)'(-r(\varphi_0))(r(\varphi)-r(\varphi_0))\\&\quad+\left(\int_0^1(\varphi-\varphi_0)'(-r(\varphi_0)+t(r(\varphi)-r(\varphi_0))\right)\\&\quad-(\varphi-\varphi_0)'(-r(\varphi_0))\,dt)(r(\varphi)-r(\varphi_0)).\end{aligned}\tag{20}$$

The integral term is of the order of $o(r(\varphi)-r(\varphi_0))$ when φ is close enough to φ_0 in C^1. On the other hand, we have

$$|r(\varphi)-r(\varphi_0)|\leqslant\|Dr(\varphi_0)\|_{\mathscr{L}(C;IR)}\|\varphi-\varphi_0\|_\infty+o(\|\varphi-\varphi_0\|_\infty),$$

so,

$$\begin{aligned}(\varphi(-r(\varphi))-\varphi_0(-r(\varphi)))&=(\varphi-\varphi_0)(-r(\varphi_0))\\&\quad-(\varphi-\varphi_0)'(-r(\varphi_0))\,Dr(\varphi_0)(\varphi-\varphi_0)\\&\quad+o(\|\varphi-\varphi_0\|_{0,1})\end{aligned}\tag{21}$$

which reads

$$(\varphi(-r(\varphi))-\varphi_0(-r(\varphi)))=(\varphi-\varphi_0)(-r(\varphi_0))+o(\|\varphi-\varphi_0\|_{0,1}).\tag{22}$$

Using now the Taylor expansion of φ_0 at $-r(\varphi_0)$, we have

$$\begin{aligned}(\varphi_0(-r(\varphi))-\varphi_0(-r(\varphi_0))&=\varphi_0'(-r(\varphi_0))(r(\varphi_0)-r(\varphi))\\&\quad+o(|r(\varphi_0)-r(\varphi)|)\\&=\varphi_0'(-r(\varphi_0))\,Dr(\varphi_0)(\varphi-\varphi_0)\\&\quad+o(\|\varphi-\varphi_0\|_{0,1}).\end{aligned}$$

Putting all these quantities together, we obtain

$$F(\varphi)-F(\varphi_0)=f'(\varphi_0(-r(\varphi_0)))\{(\varphi-\varphi_0)(-r(\varphi_0)) + \varphi_0'(-r(\varphi_0))\, Dr(\varphi_0)(\varphi-\varphi_0)\} + o(\|\varphi-\varphi_0\|_{0,1}) \tag{23}$$

which shows that F is differentiable at φ_0 in C^1 and

$$L_{\varphi_0}(\psi)=f'(\varphi_0(-r(\varphi_0)))\{\psi(-r(\varphi_0))+\varphi_0'(-r(\varphi_0))\, Dr(\varphi_0)\,\psi\}.$$

Clearly the formula shows that the map $\varphi_0 \longmapsto L_{\varphi_0}$ is continuous, which yields that the restriction of F to C^1 is of class C^1

Remark 4.3. One can improve the result of Lemma 4.2 to obtain the following result: Under the assumption H_3, for each $\varphi_0 \in C^1$, $R>0$ and $\varepsilon>0$, there exists $\gamma(\varepsilon)>0$, such that $\varphi \in C^1$, $\|\varphi-\varphi_0\|_\infty \leqslant \gamma(\varepsilon)$ and $\|(\varphi-\varphi_0)'\|_\infty \leqslant R$, imply that $|F(\varphi)-F(\varphi_0)-L_{\varphi_0}(\varphi-\varphi_0)| \leqslant \varepsilon R\, \|\varphi-\varphi_0\|_\infty$.

Proof of Proposition 4.1. Let $\varphi_0 \in \mathbf{E}_1$(where $\mathbf{E}_1$ is defined in (18)). It follows that $x=x^{\varphi_0}$ is C^1 on the interval $[-M, \infty[$. Given $t \geqslant 0$ and $\varepsilon>0$, we have

$$lip(\varphi_0)+\sup\{|x'(t)| : s \in [-M, t+\varepsilon]\}=\rho<\infty.$$

For each real number h small enough such that $|h| \leqslant \varepsilon$ and $t+h \geqslant 0$, we have

$$\frac{x'(t+h)-x'(t)}{h}=\frac{F(x_{t+h})-F(x_t)}{h} = L_{x_t}\left(\frac{x_{t+h}-x_t}{h}\right)+\frac{1}{h}\, o(\|x_{t+h}-x_t\|_{0,1}). \tag{24}$$

We will show that

$$\|x_{t+h}-x_t\|_{0,1}=O(h). \tag{25}$$

Recall that

$$\|x_{t+h}-x_t\|_{0,1}=\max\{\|x_{t+h}-x_t\|_\infty, \|x'_{t+h}-x'_t\|_\infty\}$$

and observe that

$$\|x_{t+h}-x_t\|_\infty=\sup\{|x(t+h+\theta)-x(t+\theta)| : \theta \in [-M, 0]\} \leqslant \rho\, |h|.$$

To conclude, it is then sufficient to show that there exists a constant $K \geqslant 0$ such that

$$\|x'_{t+h} - x'_t\|_\infty \leqslant K\,|h|. \tag{26}$$

From inequalities (4), (13), (14), and (15), there exists a constant $\eta > 0$ such that

$$|F(x_s) - F(x_{s'})| \leqslant \eta\,\|x_s - x_{s'}\|_\infty, \qquad \text{for each} \quad s, s' \in [0, t+1]. \tag{27}$$

Let $\theta \in [-M, 0]$.

First Case. $t+\theta \geqslant 0$. If $t+\theta+h \geqslant 0$, then

$$|x'_{t+h}(\theta) - x'_t(\theta)| = |F(x_{t+h+\theta}) - F(x_{t+\theta})|.$$

From (27), we deduce that

$$|x'_{t+h}(\theta) - x'_t(\theta)| \leqslant \eta\,\|x_{t+h+\theta} - x_{t+\theta}\|_\infty.$$

Thus,

$$|x'_{t+h}(\theta) - x'_t(\theta)| \leqslant \eta\rho\,|h|.$$

If $t+\theta+h \leqslant 0$, then $t+\theta \leqslant |h|$, and

$$\begin{aligned} |x'_{t+h}(\theta) - x'_t(\theta)| &= |\varphi'_0(t+\theta+h) - F(x_{t+\theta})| \\ &\leqslant |\varphi'_0(t+\theta+h) - \varphi'_0(0)| + |F(x_0) - F(x_{t+\theta})| \\ &\leqslant \rho\,|t+\theta+h| + \eta\,\|x_{t+\theta} - x_0\|_\infty \\ &\leqslant 2\rho\,|h| + \eta\rho\,|h|. \end{aligned} \tag{28}$$

Then we have

$$|x'_{t+h}(\theta) - x'_t(\theta)| \leqslant 3\rho\max(1, \eta)\,|h|.$$

Second Case. $t+\theta \leqslant 0$. Similarly as in the first case, we obtain

$$|x'_{t+h}(\theta) - x'_t(\theta)| \leqslant 3\rho\max(1, \eta)\,|h|.$$

Then

$$\|x'_{t+h} - x'_t\|_\infty \leqslant 3\rho\max(1, \eta)\,|h|.$$

Hence, the claimed inequality (26) holds with $K = 3\rho\max(1, \eta)$.

Finally, using Eqs. (24), (26), and Lemma 4.2, we obtain

$$(29)\quad \lim_{h\to 0}\frac{x'(t+h)-x'(t)}{h}$$
$$=\lim_{h\to 0} f'(x(t-r(x_t)))$$
$$\times\left\{\left(\frac{x_{t+h}-x_t}{h}\right)(-r(x_t))-x'(t-r(x_t))\,Dr(x_t)\left(\frac{x_{t+h}-x_t}{h}\right)\right\}$$
$$=f'(x(t-r(x_t)))\{x'(t-r(x_t))-x'(t-r(x_t))\,Dr(x_t)\,x_t'\}.$$

Since the second quantity in the right hand side of expression (28) is continuous with respect to $t\geqslant 0$, we deduce that x'' exists and is continuous at each point $t\geqslant 0$. Moreover, we have

$$x''(t)=f'(x(t-r(x_t)))\,x'(t-r(x_t))\{1-Dr(x_t)\,x_t'\},\qquad t\geqslant 0. \quad ▮$$

Corollary 4.4. *Suppose that* H_1, H_2, *and* H_3 *hold. For each* $\varphi_0\in C^{0,1}$ *the solution* x^{φ_0} *of Eq.* (2) *is* C^2 *on the interval* $[M,\infty[$. *Furthermore, the second order derivative of* x^{φ_0} *is given by formula* (19).

Proof. Let $\varphi_0\in C^{0,1}$, and $t\geqslant M$. We know that $x_t:=x_t^{\varphi_0}$ is C^1 and satisfies the condition: $x_t'(0)=F(x_t)$. Moreover, (17) implies that $x_t'\in C^{0,1}$. Then using Proposition 4.1, we conclude that $\frac{d}{dt}x^{\varphi_0}(t)$ is differentiable at each $t\geqslant M$. ▮

Corollary 4.5. *Suppose that* H_1, H_2, *and* H_3 *hold. For each* $\varphi_0\in C$, x^{φ_0} *(where* x^{φ_0} *is any solution of Eq.* (2) *with* φ_0 *as initial function) is* C^2 *on the interval* $[2M,\infty[$. *Moreover, the second order derivative of* x^{φ_0} *is given by formula* (19).

Proof. Let $\varphi_0\in C^{0,1}$, and $t\geqslant 2M$. If x^{φ_0} is a solution of (2), we have $x^{\varphi_0}(t)=x^{x_M^{\varphi_0}}(t-M)$. By Corollary 4.4 and the fact $x_M^{\varphi_0}\in C^1$, we conclude that $\frac{d}{dt}x^{\varphi_0}$ is differentiable at each $t\geqslant 2M$. ▮

5. THE INFINITESIMAL GENERATOR OF THE SEMIGROUP

In this section we characterize the infinitesimal generator of the semigroup $T(t)$, $t\geqslant 0$, that is to say, the operator A defined as

$$A\varphi=\lim_{t\searrow 0}\frac{T(t)\,\varphi-\varphi}{t}$$

when this limit exists, in $C^{0,1}$. Clearly, $A\varphi = \varphi'$. What makes A unique is its domain, that is, the set

$$D(A) = \left\{ \varphi \in \mathbf{E} : \lim_{t \searrow 0^+} \frac{T(t)\,\varphi - \varphi}{t} \text{ exists} \right\}.$$

Defining the set

$$\mathbf{E}_2 = \{\varphi \in C^2 : \varphi'^-(0) = F(\varphi) \text{ and } \varphi''^-(0) = L_\varphi(\varphi')\}$$

we have the following

PROPOSITION 5.1. *Suppose* H_1, H_2, *and* H_3 *hold. Then,*

$$D(A) = \mathbf{E}_2. \tag{30}$$

Proof. Observe that $\varphi \in D(A)$ if and only if there exists $\psi \in C^{0,1}$ such that

$$\lim_{t \searrow 0^+} \left\| \frac{T(t)\,\varphi - \varphi}{t} - \psi \right\|_\infty = 0 \tag{31}$$

and

$$\lim_{t \searrow 0^+} \left\| \frac{d}{d\theta} \left(\frac{T(t)\,\varphi - \varphi}{t} - \psi \right) \right\|_{L^\infty} = 0,$$

that is,

$$\lim_{t \searrow 0^+} \left\| \frac{x'(t+\cdot) + x'(\cdot)}{t} - \psi' \right\|_{L^\infty} = 0, \tag{32}$$

where $x' = \frac{d}{dt} x^\varphi$. We know that (31) is equivalent to $\varphi \in C^1, \varphi' = \psi$ and $\varphi'(0) = F(\varphi)$ (see [14, Proposition 3.1]).

We start by showing that each element of $D(A)$ is of class C^2. Let $\varphi \in D(A)$. Set $x = x^\varphi$; let $(t_n)_{n \geqslant 0}$ be a decreasing sequence of positive numbers, with $\lim_{n \to \infty} t_n = 0$, and denote $\zeta_n = (x'(t_n + \cdot) - x'(\cdot))/t_n$. From (32) we deduce that (ζ_n) is a Cauchy sequence in L^∞. We know from Proposition 3.3 that the solutions starting in $\mathbf{E}$ are C^1 on their domain. Denote $\zeta = \lim_{n \to \infty} \zeta_n$. Since ζ_n converges almost a.e to φ'' (φ'' is the derivative of φ'), then $\varphi'' = \zeta$ So, φ'' is continuous. Proceeding as in the proof of Proposition 3.3 (see (8)) we deduce $\varphi' \in C^1$, so φ is C^2.

We now prove that $\varphi'^{-}(0) = L_{\varphi}(\varphi')$.

The functions in formula (32) are continuous, thus the converence holds in C, that is, we can write

$$\lim_{t \searrow 0^+} \left\| \frac{x'(t+\cdot) - x'(\cdot)}{t} - \psi' \right\|_{\infty} = 0,$$

in particular, we have

$$\lim_{t \searrow 0^+} \left| \frac{x'(t) - x'(0)}{t} - \varphi''(0) \right| = 0. \tag{33}$$

By using Proposition 4.1, Lemma 4.2, and (33) we deduce that

$$\varphi''^{-}(0) = L_{\varphi}(\varphi').$$

Thus, we have proved that

$$D(A) \subset \mathbf{E}_2.$$

Conversely, let $\varphi \in C^2$ be such that $\varphi^{-}(0) = F(\varphi)$, and $\varphi''(0) = L_{\varphi}(\varphi')$. Proposition 4.1 implies that the solution x of the equation (2) is twice continuously differentiable on the interval $[-M, \infty[$. One deduces that (30) and (31) are satisfied with $\psi = \varphi'$. This completes the proof of the proposition. ▮

COROLLARY 5.2. *Suppose* H_1, H_2 *and* H_3 *hold. If we choose an initial datum* $\varphi \in D(A)$, *then the solution of* (2) x^{φ} *is* C^2 *on the interval* $[-M, \infty[$.

COROLLARY 5.3. *Suppose* H_1, H_2, *and* H_3 *hold. Then, we have:*

(a) $T(t)(\mathbf{E}_2) \subseteq \mathbf{E}_2$, *for each* $t \geqslant 0$.

(b) $T(t)(C^{0,1}) \subseteq \mathbf{E}_2$ *for each* $t \geqslant 2M$.

From Proposition 4.1, we can deduce the following result.

PROPOSITION 5.4. *Suppose* H_1, H_2, *and* H_3 *hold. Then, the closure of the domain* $\mathbf{E}_2$ *in the space* $(C^{0,1}, \|\cdot\|_{0,1})$ *is the set* $\mathbf{E}$.

The proof of Proposition 5.4 hinges on two auxiliary results.

Let us first introduce further notations:

$$C_0 = \{\varphi \in C^1 : \varphi(0) = 0\}$$

C_0^1 is the subspace of C^1, defined by

$$C_0^1 = \{\varphi \in C^1 : \varphi, \varphi' \in C_0\}, \tag{34}$$

A_0 is the operator defined in C_0^1 by

$$\begin{aligned} D(A_0) &= \{\varphi \in C_0^1 : \varphi' \in C_0^1\} \\ A_0(\varphi) &= \varphi'. \end{aligned} \tag{35}$$

LEMMA 5.5. *For each $\varphi \in C_0^1$, we have*

$$\lim_{\lambda \to 0} \|(I - \lambda A_0)^{-1} \varphi - \varphi\|_1 = 0, \tag{36}$$

where $\|\cdot\|_1$ is the norm of the space C^1, defined by

$$\|\varphi\|_1 = \max\{\|\varphi\|_\infty, \|\varphi'\|_\infty\}$$

Proof of Lemma 5.5. It is known (see, for example, [14]) that

$$\begin{cases} y'(t) = 0 \\ y_0 = \varphi \end{cases}$$

determines on C (first) and on C_0 (by restriction) a C_0-semigroup $T_0(t)$, which has the operator B_0 defined by

$$\begin{cases} D(B_0) = \{\varphi \in C_0 : \varphi' \in C_0\} \\ B_0\phi = \phi', \end{cases}$$

as an infinitesimal generator. On the other hand, the operator

$$\begin{aligned} \mathscr{I} : C_0 &\longrightarrow C_0^1 \\ \varphi &\longmapsto \int_\cdot^0 \varphi(s)\, ds \end{aligned}$$

is an isomorphism between $(C_0, \|\cdot\|_\infty)$ and $(C_0^1, \|\cdot\|_1)$. It is not difficult to see that the family of operators defined by

$$S(t) = \mathscr{I} \circ T_0(t) \circ \mathscr{I}^{-1}, \qquad \text{for each} \quad t \geqslant 0,$$

is an C_0-semigroup. We prove that $S(t)$ has A_0 (the operator defined on C_0^1 by (35)) as an infinitesimal generator. Let $\psi \in C_0^1$. Then $\lim_{t \searrow 0^+}((S(t)\,\psi - \psi)/t)$

exists in $(C_0^1, \|\cdot\|_1)$ if and only if $\lim_{t\searrow 0^+}((T_0(t)\circ \mathscr{J}^{-1}\psi - \mathscr{J}^{-1}\psi)/t)$ exists in $(C_0, \|\cdot\|_\infty)$, i.e. : $\mathscr{J}^{-1}\psi \in D(B_0)$. Since, $\mathscr{J}^{-1}\psi = -\psi'$, we deduce that ψ is in the domain of the infinitesimal generator of $S(t)$ if and only if $\psi \in D(A_0)$ and $\lim_{t\searrow 0^+}((S(t)\,\psi - \psi)/t) = \mathscr{J}(B_0\mathscr{J}^{-1}\psi) = \psi'$. We deduce that A_0 is the infinitesimal generator of the C_0-semigroup $S(t)$. The result follows from the Hille–Yoshida theorem (see, for example, [9]). ▮

We now introduce a function χ defined on $]-\infty, 0]$ with values in $[0, 1]$, and satisfying the following properties

$$\begin{aligned} &\text{(i)} && \chi \text{ is } C^2,\\ &\text{(ii)} && \chi(s) = 0 \text{ if } s \notin [-1, 0],\\ &\text{(iii)} && \chi(s) \leqslant 1,\\ &\text{(iv)} && \chi(0) = 1,\\ &\text{(v)} && \chi'(0) = 0. \end{aligned} \tag{37}$$

LEMMA 5.6. *If χ satisfies conditions* (i)–(v) *of* (37), *then*

(a) *The function*

$$\begin{aligned} \Psi_\varepsilon : [-M, 0] &\longrightarrow IR\\ \theta &\longmapsto \frac{\theta}{\varepsilon}\chi\left(\frac{\theta}{\varepsilon}\right), \end{aligned} \tag{38}$$

is bounded independently of $\varepsilon > 0$.

(b) *The function $\Gamma_{(a,b,\varepsilon)}$ of $C^1([-M, 0], IR)$ defined, for all $(a, b, \varepsilon) \in IR \times IR \times IR_+^*$ by*

$$\Gamma_{(a,b,\varepsilon)}(\theta) = a\theta\chi\left(\frac{\theta}{\varepsilon}\right) + \frac{1}{2}b\frac{\theta^2}{\varepsilon}\chi\left(\frac{\theta}{\varepsilon^2}\right), \qquad \textit{for all} \quad \theta \in [-M, 0],$$

converges to zero in the space $(C^1, \|\cdot\|_1)$, as (ε, a, b) tends to $(0, 0, 0)$.

Proof of Lemma 5.6. (a) Let $\varepsilon > 0, \theta \in [-M, 0]$. If $\frac{\theta}{\varepsilon} \leqslant -1$, then $\chi(\frac{\theta}{\varepsilon}) = 0$. If $\frac{\theta}{\varepsilon} \in [-1, 0]$, then

$$\left|\frac{\theta}{\varepsilon}\chi\left(\frac{\theta}{\varepsilon}\right)\right| \leqslant \left|\chi\left(\frac{\theta}{\varepsilon}\right)\right| \leqslant 1.$$

So, we deduce that $\|\Psi_\varepsilon\|_\infty \leqslant 1$, for each $\varepsilon > 0$.

(b) Notice that Γ is, for each fixed value $(a, b, \varepsilon) \in IR \times IR \times IR_*^+$, of class C^1 on $[-M, 0]$. We will evaluate $\|\Gamma_{(a,b,\varepsilon)}(\cdot)\|_1$. From (a), we have

$$|\Gamma_{(a,b,\varepsilon)}(\theta)| \leqslant \varepsilon\,|a| \left|\frac{\theta}{\varepsilon}\,\chi\left(\frac{\theta}{\varepsilon}\right)\right| + \frac{1}{2}\,\varepsilon\,|b\theta| \left|\frac{\theta}{\varepsilon^2}\,\chi\left(\frac{\theta}{\varepsilon^2}\right)\right| \tag{39}$$

$$\leqslant \varepsilon\,|a| + \frac{1}{2}\,\varepsilon\,|b|\,M.$$

and

$$\left|\frac{d}{d\theta}\,\Gamma_{(a,b,\varepsilon)}(\theta)\right| = \left|a\chi\left(\frac{\theta}{\varepsilon}\right) + a\,\frac{\theta}{\varepsilon}\,\chi'\left(\frac{\theta}{\varepsilon}\right) + b\,\frac{\theta}{\varepsilon}\,\chi\left(\frac{\theta}{\varepsilon^2}\right) + \frac{1}{2}\,b\,\frac{\theta^2}{\varepsilon^3}\,\chi'\left(\frac{\theta}{\varepsilon^2}\right)\right| \tag{40}$$

$$\leqslant |a| + \varepsilon\,|b| + |a| \left|\frac{\theta}{\varepsilon}\,\chi'\left(\frac{\theta}{\varepsilon}\right)\right| + \frac{1}{2}\,\varepsilon\,|b| \left|\frac{\theta}{\varepsilon^2}\,\chi'\left(\frac{\theta}{\varepsilon^2}\right)\right| \left|\frac{\theta}{\varepsilon^2}\right|.$$

In the same way as in (a), one can show that the function $\theta \longmapsto |\frac{\theta}{\varepsilon}\,\chi'(\frac{\theta}{\varepsilon})|$ is bounded independently of $\varepsilon > 0$. We have

$$\left|\frac{\theta}{\varepsilon}\,\chi'\left(\frac{\theta}{\varepsilon}\right)\right| \leqslant \|\chi'\|_\infty.$$

We also have

$$\left|\frac{\theta}{\varepsilon^2}\,\chi'\left(\frac{\theta}{\varepsilon^2}\right)\right| \leqslant \|\chi'\|_\infty.$$

Moreover,

$$\chi'\left(\frac{\theta}{\varepsilon^2}\right) = 0 \qquad \text{for} \quad |\theta| \geqslant \varepsilon^2.$$

Thus, we deduce that

$$\left|\frac{d}{d\theta}\,\Gamma_{(a,b,\varepsilon)}\,(\theta)\right| \leqslant |a| + \varepsilon\,|b| + |a| \sup_{IR}\,|\chi'| + \frac{\varepsilon}{2}\,|b| \sup_{IR}\,|\chi'|, \tag{41}$$

and we have the convergence of Γ to 0 in C^1, as $(a, b, \varepsilon) \to 0$.

Proof of the Proposition 5.4. Let $\phi \in \mathbf{E}$. Our goal here is to approximate this function in $(C^{0,1}, \|\cdot\|_{0,1})$, by a sequence of functions in $\mathbf{E}_2$.

For each $\varepsilon > 0$, $a \in IR$, and $b \in IR$, we define the functions ϕ_ε and $\phi_{\varepsilon,a,b}$ by

$$(42)\qquad \begin{aligned} \phi_\varepsilon(\theta) &= (I - \varepsilon A_0)^{-1}(\phi_0)(\theta) + \theta\phi'(0) + \phi(0), \\ \phi_{\varepsilon,a,b}(\theta) &= \phi_\varepsilon(\theta) + \Gamma_{(a,b,\varepsilon)}(\theta), \qquad \theta \in [-M, 0], \end{aligned}$$

where $\phi_0(\theta) = \phi(\theta) - \theta\phi'(0) - \phi(0)$, $\theta \in [-M, 0]$. Lemmas 5.5 and 5.6 imply that

$$(43)\qquad \lim_{\varepsilon, a, b \to 0} \|\phi - \phi_{\varepsilon,a,b}\|_1 = 0.$$

Given $\xi > 0$. From property (43), there exist $\varepsilon_1 = \varepsilon_1(\xi) > 0$, $a_1 = a_1(\xi) > 0$, and $b_1 = b_1(\xi) > 0$, such that

$$(44)\qquad \|\phi - \phi_{\varepsilon,a,b}\|_1 \leqslant \xi, \qquad \text{for each} \quad (\varepsilon, a, b) \in \mathbf{B}_1,$$

where

$$\mathbf{B}_1 =]0, \varepsilon_1] \times [-a_1, a_1] \times [-b_1, b_1].$$

So, it is sufficient to determine$(\varepsilon, a, b) \in \mathbf{B}_1$, such that $\phi_{\varepsilon,a,b} \in \mathbf{E}_2$. Observe that the functions ϕ_ε and $\phi_{\varepsilon,a,b}$ are C^2 and satisfy $\phi_\varepsilon(0) = \phi(0)$, $\phi'_\varepsilon(0) = \phi'(0)$, $\phi''_\varepsilon(0) = 0$, $(d/d\theta)\,\phi_{\varepsilon,a,b}(0) = a + \phi'(0)$ and $(d^2/d\theta^2)\,\phi_{\varepsilon,a,b}(0) = b/\varepsilon$. This implies that $\phi_{\varepsilon,a,b} \in \mathbf{E}_2$ if and only if (i) and (ii) hold at the same time where

(i) $\phi'(0) + a = F(\phi_{\varepsilon,a,b})$

(ii) $\frac{b}{\varepsilon} = L_{\phi_{\varepsilon,a,b}}(\frac{d}{d\theta}\,\phi_{\varepsilon,a,b})$.

The end of the proof is done in two parts. First, we look for the elements of the set $\mathbf{B}_1$ which satisfy (i). Second we show that amongst these elements there exists at least one element for which (ii) holds.

Claim 1. There exist $0 < \bar{b} < b_1$, $0 < \bar{\varepsilon} < \varepsilon_1$, such that for each $(\varepsilon, b) \in \bar{\mathbf{B}}_1 =]0, \bar{\varepsilon}] \times [-\bar{b}, \bar{b}]$. Equation (i) has at least one solution a.

We have to solve equation $G(\varepsilon, a, b) = a$, $(\varepsilon, a, b) \in \mathbf{B}_1$, where G is a function defined from $\mathbf{B}_1$ into IR by $G(\varepsilon, a, b) = F(\phi_{\varepsilon,a,b}) - \phi'(0)$.

We now consider the sequence of functions defined by

$$\begin{aligned} a_n(\varepsilon, b) &= G(\varepsilon, a_{n-1}(\varepsilon, b), b) \\ a_0(\varepsilon, b) &= 0. \end{aligned}$$

We show that there exists $(\varepsilon', b') \in]0, \varepsilon_1] \times]0, b_1]$ such that the sequence of functions $(a_n(\varepsilon, b))_{n \geqslant 1}$ converges to a function $\tilde{a}(\varepsilon, b)$ which is continuous in b, on the set $]0, \varepsilon'] \times [-b', b']$. We have

$$\lim_{\varepsilon \to 0} \sup \left\{ \left| \frac{\partial}{\partial a} G(\varepsilon, a, b) \right|; \ a \in [-a_1, a_1] \text{ et } b \in [-b_1, b_1] \right\} = 0. \tag{45}$$

In fact, Lemma 4.2 implies that the function G is differentiable with respect to a and

$$\frac{\partial}{\partial a} G(\varepsilon, a, b) = L_{\phi_{\varepsilon, a, b}}\left((\cdot)\, \chi\left(\frac{\cdot}{\varepsilon}\right)\right). \tag{46}$$

Lemmas 4.2 and 5.6 and inequalities (4), (44) imply that for λ small enough

$$\frac{\left| F\left(\phi_{\varepsilon, a, b} + \lambda (\cdot)\, \chi\left(\frac{\cdot}{\varepsilon}\right)\right) - F(\phi_{\varepsilon, a, b})\right|}{\lambda} \leqslant Q \left\| (\cdot)\, \chi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{\infty} \tag{47}$$

$$\leqslant \varepsilon Q, \qquad \text{for each} \quad (\varepsilon, a, b) \in \mathbf{B}_1, \tag{48}$$

where $Q = lip_{\gamma}(f)\{lip_{\gamma}(r)\, \gamma + 1\}$, and $\gamma = (\|\phi\|_1 + 1)$. Taking (46) into account, we deduce

$$\left| \frac{\partial}{\partial a} G(\varepsilon, a, b) \right| = \lim_{\lambda \to 0} \frac{\left| F\left(\phi_{\varepsilon, a, b} + \lambda (\cdot)\, \chi\left(\frac{\cdot}{\varepsilon}\right)\right) - F(\phi_{\varepsilon, a, b})\right|}{\lambda} \tag{49}$$

$$\leqslant \varepsilon Q, \qquad \forall (\varepsilon, a, b) \in \mathbf{B}_1. \tag{50}$$

This proves (45).

Using the above results we obtain the existence of $(\varepsilon'', b'') \in]0, \varepsilon_1] \times [-b_1, b_1]$ such that

(51)

$$|G(\varepsilon, a, b)| \leqslant a_1, \qquad \text{for each} \quad (\varepsilon, a, b) \in]0, \varepsilon''] \times [-a_1, a_1] \times [-b'', b''].$$

From (45), there exists $\varepsilon_2 \in]0, \varepsilon_1]$ such that

$$\sup \left\{ \frac{\partial}{\partial a} G(\varepsilon, a, b); \ a \in [-a_1, a_1] \text{ and } b \in [-b_1, b_1] \right\} \tag{52}$$

$$\leqslant \frac{1}{2}, \qquad \text{for each} \quad \varepsilon \in]0, \varepsilon_2].$$

Equation (43) and the fact that $\phi \in \mathbf{E}$, imply that

$$\lim_{(\varepsilon, a, b) \to (0, 0, 0)} G(\varepsilon, a, b) = 0.$$

Then, we deduce the existence of $(\varepsilon_3, b_2) \in]0, \varepsilon_2] \times]0, b_1]$, such that

$$|G(\varepsilon, 0, b)| \leqslant \frac{a_1}{2}, \qquad \text{for each} \quad (\varepsilon, b) \in]0, \varepsilon_3] \times]0, b_2]. \tag{53}$$

So, by combining (52), and (53), we obtain (51), with $\varepsilon'' = \varepsilon_3$ and $b'' = b_2$.
From inequalities (51) and (52), we have

$$\begin{cases} (a_n(\varepsilon, b))_{n \geqslant 0} \subset [-a_1, a_1], & \text{for each} \quad (\varepsilon, b) \in]0, \varepsilon_3] \times [-b_2, b_2] = \widetilde{\mathbf{B}}_2, \\ |a_n(\varepsilon, b) - a_m(\varepsilon, b)| \leqslant \dfrac{1}{2^m} a_1, & \text{for each} \quad n \geqslant m \geqslant 1, (\varepsilon, b) \in \widetilde{\mathbf{B}}_2. \end{cases}$$

Since the function $G(\varepsilon, a, b)$ is continuous in (a, b), and the functions $a_n(\varepsilon, b)$, $n \geqslant 1$, are continuous in b, then the sequence $(a_n(\varepsilon, b))_{n \geqslant 0}$ converges uniformly on the set $\widetilde{\mathbf{B}}_2$, to a function $\tilde{a}$ defined from $\widetilde{\mathbf{B}}_2$ into $[-a_1, a_1]$, continuous in b, and satisfying $\tilde{a}(\varepsilon, b) = G(\varepsilon, \tilde{a}(\varepsilon, b), b)$, for each $(\varepsilon, b) \in \widetilde{\mathbf{B}}_2$. Thus Claim 1 holds with $\bar{\varepsilon} = \varepsilon_3$, $\bar{b} = b_2$, $\bar{\mathbf{B}}_1 = \widetilde{\mathbf{B}}_2$.

Claim 2. There exists $0 < \bar{\bar{\varepsilon}} < \bar{\varepsilon}$ such that if we denote $V_{\varepsilon, a}(b) = \varepsilon L_{\phi_{\varepsilon, a, b}}(\frac{d}{d\theta} \phi_{\varepsilon, a, b})$, $a = \tilde{a}(\varepsilon, b)$ then equation $V_{\varepsilon, a}(b) = b$ has at least one a solution for each $\varepsilon \in]0, \bar{\bar{\varepsilon}}]$

Using the same arguments as in (47) and (50), we can show that there exists a positive constant Q such that

$$|V_{\varepsilon, a}(b)| \leqslant \varepsilon Q \left\| \frac{d}{d\theta} \phi_{\varepsilon, a, b} \right\|_\infty. \tag{54}$$

By differentiating in (42) we have

$$\frac{d}{d\theta} \phi_{\varepsilon, a, b}(\theta) = \frac{d}{d\theta} (I - \varepsilon A_0)^{-1} (\phi_0)(\theta) + \phi'(0) + \frac{d}{d\theta} \Gamma_{(a, b, \varepsilon)}(\theta). \tag{55}$$

From (41) we obtain

$$\varepsilon \left| \frac{d}{d\theta} \Gamma_{(a, b, \varepsilon)}(\theta) \right| \leqslant \varepsilon \left(a_1 + \varepsilon b_2 + a_1 \sup_{IR} |\chi'| + \frac{\varepsilon}{2} b_2 \sup_{IR} |\chi'| \right), \tag{56}$$

$$\text{for each} \quad (\varepsilon, a, b) \in]0, \varepsilon_3] \times [-a_1, a_1] \times [-b_2, b_2].$$

By using (54), (56), (55), and Lemma 5.5 we deduce that

$$\lim_{\varepsilon \to 0} \sup \{|V_{\varepsilon, a}(b)| : (a, b) \in [-a_1, a_1] \times [-b_2, b_2]\} = 0.$$

Therefore, there exists $\varepsilon_4 \in]0, \varepsilon_3]$ such that, for each $0 < \varepsilon \leqslant \varepsilon_4$, we have

$$\begin{cases} |V_{\varepsilon, a}(b)| \leqslant b_2, & (a, b) \in [-a_1, a_1] \times [-b_2, b_2], \\ |V_{\varepsilon, \tilde{a}(\varepsilon, b)}(b)| \leqslant b_2, & b \in [-b_2, b_2]. \end{cases}$$

We conclude that for each fixed $\varepsilon \in]0, \varepsilon_4]$, the function $V_{\varepsilon, \tilde{a}(\varepsilon, \cdot)}(\cdot)\colon [-b_2, b_2] \to [-b_2, b_2]$ is continuous and has a fixed point $b(\varepsilon)$ in the interval $[-b_2, b_2]$.
Then the proof of Claim 2 is complete.

To summarize, the values $a = \tilde{a}(\varepsilon, b(\varepsilon))$ and $b = b(\varepsilon)$ determined in Claim 1 and Claim 2, respectively, are such that

$$\phi_{\varepsilon, a, b} \in \mathbf{E}_2 \qquad \text{and} \qquad \|\phi_{\varepsilon, a, b} - \phi\|_1 \leqslant \xi.$$

This completes the proof of Proposition 5.4. ∎

6. APPROXIMATION OF THE SEMIGROUP

In this section we establish an approximation result of the semigroup solution, $T(t)$, $t \geqslant 0$ based on the Crandall–Liggett theorem. Our method uses a technique developed in the case of neutral delay equations by Plant [11].

In the sequel, for each $k > 0$ and $\gamma > 0$, we denote $\|\cdot\|_{1,\gamma}$, the norm on C^1, equivalent to $\|\cdot\|_1$, defined by

$$\|\varphi\|_{1,\gamma} = \|\varphi\|_\infty + \frac{1}{\gamma} \|\varphi'\|_\infty, \qquad \varphi \in C^1, \tag{57}$$

$B_\gamma(k)$ is the ball of centre 0 and radius $k > 0$ of C^1, endowed with the norm $\|\cdot\|_{1,\gamma}$. Define the following function

$$V_\gamma(\varphi) = \begin{cases} \varphi & \text{if } \|\varphi\|_{1,\gamma} \leqslant 1, \\ \dfrac{\varphi}{\|\varphi\|_{1,\gamma}} & \text{if } \|\varphi\|_{1,\gamma} \geqslant 1. \end{cases} \tag{58}$$

V_γ is a retraction on the ball of center 0 and radius 1, with respect to the norm $\|\cdot\|_{1,\gamma}$, We also define the retraction to the ball of center 0 and radius 1, with respect to the norm $\|\cdot\|_\infty$

$$V(\varphi)=\begin{cases}\varphi & \text{if } \|\varphi\|_\infty \leqslant 1,\\ \dfrac{\varphi}{\|\varphi\|_\infty} & \text{if } \|\varphi\|_\infty \geqslant 1.\end{cases}$$

$F_{\gamma,k}$ is the function defined on C^1 by

(59)

$$F_{\gamma,k}(\varphi)=f\left(k\left(V_\gamma\left(\frac{1}{k}\varphi\right)\right)\left(-r\left(k\left(V\left(\frac{1}{k}\varphi\right)\right)\right)\right)\right), \qquad \text{for each} \quad \varphi\in C^1$$

We denote by E^k the subset of C^1 defined by

$$E^k=\{\varphi\in C^1 : \varphi'^-(0)=F_k(\varphi)\}, \tag{60}$$

where

$$F_k=F_{1+lip_k(f),k}. \tag{61}$$

A^k is the operator defined on C^1 by

$$\begin{aligned} &D(A^k)=\{\varphi\in C^2 : \varphi'^-(0)=F_k(\varphi)\}\\ &A^k\varphi=\varphi'. \end{aligned} \tag{62}$$

Finally, A_1 is the operator defined by

$$\begin{aligned} &D(A_1)=\{\varphi\in C^2 : \varphi'^-(0)=F(\varphi)\},\\ &A_1\varphi=\varphi'. \end{aligned} \tag{63}$$

LEMMA 6.1. *Suppose* H_1 *and* H_2 *be satisfied. Then for each real* $k>0$ *and* $\gamma>0$, *there exists a continuous function* $F_{\gamma,k}: C^1\to IR$, *such that*

(a) $F(\psi)=F_{\gamma,k}(\psi)$ *for each* $\psi\in B_\gamma(k)$,

(b)

$$|F_{\gamma,k}(\psi)-F_{\gamma,k}(\varphi)|\leqslant \rho\,\|\psi-\varphi\|_\infty+\frac{lip_k(f)}{\gamma}\|\psi'-\varphi'\|_\infty \qquad \textit{for each} \quad \psi,\varphi\in C^1,$$

where $\rho=\rho(\gamma,k)=2\,lip_k(f)(\gamma k\,lip_k(r)+1)$.

Proof. Let $\gamma > 0$ and $k > 0$. We have

$$\|V_\gamma(\varphi) - V_\gamma(\phi)\|_\infty \leqslant \begin{cases} \dfrac{1}{\sup\{\|\varphi\|_{1,\gamma},\ \|\psi\|_{1,\gamma}\}}\left(2\,\|\psi-\varphi\|_\infty + \dfrac{1}{\gamma}\,\|\psi'-\varphi'\|_\infty\right) \\ \qquad \text{if} \quad \|\varphi\|_{1,\gamma} \geqslant 1 \quad \text{or} \quad \|\psi\|_{1,\gamma} \geqslant 1, \\ \|\psi-\varphi\|_\infty \quad \text{if} \quad \|\varphi\|_{1,\gamma} \leqslant 1 \quad \text{and} \quad \|\psi\|_{1,\gamma} \leqslant 1. \end{cases} \tag{64}$$

The case where $\|\varphi\|_{1,\gamma} \leqslant 1$ and $\|\psi\|_{1,\gamma} \leqslant 1$ is evident. It remains to discuss three possible cases :

First Case. If $\|\varphi\|_{1,\gamma} \geqslant 1$ and $\|\psi\|_{1,\gamma} \geqslant 1$, we have

$$\begin{aligned} \|V_\gamma(\varphi) - V_\gamma(\psi)\|_\infty &= \left\| \frac{\varphi}{\|\varphi\|_{1,\gamma}} - \frac{\psi}{\|\psi\|_{1,\gamma}} \right\|_\infty \\ &\leqslant \frac{1}{\|\varphi\|_{1,\gamma}\,\|\psi\|_{1,\gamma}}\,\{\|\psi\|_{1,\gamma}\,\|\varphi-\psi\|_\infty + |\,\|\psi\|_{1,\gamma} - \|\varphi\|_{1,\gamma}|\,\|\psi\|_\infty\} \\ &\leqslant \frac{1}{\|\varphi\|_{1,\gamma}}\left(2\,\|\psi-\varphi\|_\infty + \frac{1}{\gamma}\,\|\psi'-\varphi'\|_\infty\right). \end{aligned}$$

Second Case. If $\|\varphi\|_{1,\gamma} \leqslant 1$ and $\|\psi\|_{1,\gamma} \geqslant 1$, we obtain

$$\begin{aligned} \|V_\gamma(\varphi) - V_\gamma(\psi)\|_\infty &= \left\| \varphi - \frac{\psi}{\|\psi\|_{1,\gamma}} \right\|_\infty \\ &\leqslant \frac{1}{\|\psi\|_{1,\gamma}}\,\{|\,\|\psi\|_{1,\gamma} - 1|\,\|\varphi\|_\infty + \|\varphi-\psi\|_\infty\} \\ &\leqslant \frac{1}{\|\psi\|_{1,\gamma}}\,\{|\,\|\psi\|_{1,\gamma} - \|\varphi\|_{1,\gamma}|\,\|\varphi\|_\infty + \|\varphi-\psi\|_\infty\} \\ &\leqslant \frac{1}{\|\psi\|_{1,\gamma}}\,\{\|\psi-\varphi\|_{1,\gamma} + \|\varphi-\psi\|_\infty\} \\ &\leqslant \frac{1}{\|\psi\|_{1,\gamma}}\left(2\,\|\psi-\varphi\|_\infty + \frac{1}{\gamma}\,\|\psi'-\varphi'\|_\infty\right). \end{aligned}$$

By the same arguments we show that (64) holds true if $\|\varphi\|_{1,\gamma} \geqslant 1$ and $\|\psi\|_{1,\gamma} \leqslant 1$. From inequality (64), we deduce that

$$\begin{aligned} &\left\| V_\gamma\left(\frac{1}{k}\varphi\right) - V_\gamma\left(\frac{1}{k}\psi\right) \right\|_\infty \\ &\quad \leqslant \frac{2}{k}\,\|\psi-\varphi\|_\infty + \frac{1}{\gamma k}\,\|\psi'-\varphi'\|_\infty, \qquad \text{for each} \quad \psi, \varphi \in C^1, k > 0. \end{aligned} \tag{65}$$

On the other hand, we know from a result obtained in [10] that the function V satisfies

$$\|V(\varphi)-V(\psi)\|_\infty \leqslant 2\,\|\varphi-\psi\|_\infty.$$

The function $F_{\gamma,k}$, defined by (59), is the same as the function in Lemma 6.1. In fact, for each $\varphi \in B_\gamma(k)$, we have $kV_\gamma(\frac{1}{k}\varphi)=\varphi$ and $kV(\frac{1}{k}\varphi)=\varphi$. Then, $F(\varphi)=F_{\gamma,k}(\varphi)$. Furthermore, using (65), we have

$$\begin{aligned}
&|F_{\gamma,k}(\varphi)-F_{\gamma,k}(\psi)|\\
&\quad\leqslant lip_k(f)\,k\left|\left(V_k\left(\frac{1}{k}\varphi\right)\right)\left(-r\left(kV\left(\frac{1}{k}\varphi\right)\right)\right)\right.\\
&\qquad\left.-\left(V_k\left(\frac{1}{k}\psi\right)\right)\left(-r(kV\left(\frac{1}{k}\psi\right)\right)\right)\right|\\
&\quad\leqslant lip_k(f)\,k\left|V_k\left(\frac{1}{k}\varphi\right)\right)\left(-r\left(kV\left(\frac{1}{k}\varphi\right)\right)\right)\\
&\qquad\left.-\left(V_k\left(\frac{1}{k}\varphi\right)\right)\left(-r\left(kV\left(\frac{1}{k}\psi\right)\right)\right)\right|\\
&\qquad+lip_k(f)\,k\left|\left(V_k\left(\frac{1}{k}\varphi\right)\right)\left(-r\left(kV\left(\frac{1}{k}\psi\right)\right)\right)\right.\\
&\qquad\left.-\left(V_k\left(\frac{1}{k}\psi\right)\right)\left(-r\left(kV\left(\frac{1}{k}\psi\right)\right)\right)\right|\\
&\quad\leqslant lip_k(f)\,k\{2\gamma\, lip_k(r)\,\|\psi-\varphi\|_\infty\}\\
&\qquad+lip_k(f)\left\{2\,\|\psi-\varphi\|_\infty+\frac{1}{\gamma}\,\|\psi'-\varphi'\|_\infty\right\}\\
&\quad\leqslant lip_k(f)\{2\gamma k\, lip_k(r)+2\}\,\|\psi-\varphi\|_\infty+\frac{lip_k(f)}{\gamma}\,\|\psi'-\varphi'\|_\infty,
\end{aligned}$$

for all $\varphi, \psi \in C^1$. ∎

THEOREM 6.2 (11). *Suppose that there exist constants $\gamma>0$, $\sigma \geqslant 0$, and $0 \leqslant \gamma_\sigma < 1$ such that*

$$\begin{aligned}
&|F(\varphi)-F(\psi)|\\
&\quad\leqslant \gamma\,\|\varphi-\psi\|_\infty+\gamma_\sigma \sup_{\theta\in[-M,0]}\{exp(-\sigma\theta)\,|\varphi'(\theta)-\psi'(\theta)|\}, \qquad \textit{for each} \quad \varphi, \phi \in C^1.
\end{aligned}$$

Then the operator A_1 generates a shift semigroup $T_1(t)$ in the sense of Theorem 2.6, on the set $\mathbf{E}$. *Moreover, the function $x(t;\varphi)$ defined, for each $\varphi \in \mathbf{E}$, by*

$$x(t;\varphi)=\begin{cases}\varphi(t) & \text{if } -M \leqslant t \leqslant 0\\ (T_1(t)\,\varphi)(0) & \text{if } t>0,\end{cases}$$

is the solution of Eq. (2).

THEOREM 6.3. *Suppose that* H_1 *and* H_2 *hold. For every* $\varphi \in \mathbf{E}$ *and* $t_0>0$, $k>0$ *such that* $\|\varphi\|_{1,\, lip_k(f)+1} \leqslant k$, *we have*

$$\lim_{n\to\infty}\left\|\left(Id-\frac{t}{n}A^b\right)^{-n}\varphi-T(t)\,\varphi\right\|_1=0, \qquad \textit{for each} \quad t\in[0,t_0],$$

where $b=b(t_0,k)=(\alpha/(lip_k(f)+1)+1)(\beta t_0+k)\,exp(\alpha t_0)+(\beta+k)/(lip_k(f)+1)$.

Proof. Let $k>0$, $t_0>0$, and $\varphi\in\mathbf{E}$, such that $\|\varphi\|_{1,\, lip_k(f)+1}\leqslant k$. Denote by b, the number $b(t_0,k)$ given in Theorem 6.3. Observe that $\varphi\in E^b$. From Lemma 6.1, the function F_b satisfies the conditions of Theorem 6.2. Then the operator A^b generates a shift semigroup $T^b(t)$, in the sense of Theorem 2.6, on the set $\mathbf{E}$. Moreover, the function

$$y(t)=\begin{cases}(T^b(t)\,\varphi)(0) & \text{if } t\geqslant 0\\ \varphi(t) & \text{if } -M\leqslant t\leqslant 0,\end{cases}$$

satisfies

$$\begin{cases} y'(t)=F_b(y(t)), & t\geqslant 0\\ y_0=\varphi.\end{cases} \tag{66}$$

If $x=x^\varphi$ is the solution of Eq. (2), then

$$|x(t)|\leqslant|x(0)|+\int_0^t |F(x_v)|\,dv. \tag{67}$$

From inequalities (14) and (15) we obtain

$$\frac{1}{lip_k(f)+1}\|x_t'\|_\infty\leqslant\frac{\alpha}{lip_k(f)+1}(\beta t_0+k)\,exp(\alpha t_0)+\frac{\beta+\|\varphi'\|_\infty}{lip_k(f)+1}. \tag{68}$$

We deduce that for each $t \in [0, t_0]$

$$(69) \qquad \|x_t\|_{1, lip_k(f)+1} \leqslant \left(\frac{\alpha}{lip_k(f)+1}+1\right)(\beta t_0+k)\, exp(\alpha t_0)+\frac{\beta+k}{lip_k(f)+1}$$
$$= b.$$

Lemma 6.1 and inequality (69) imply

$$(70) \qquad \begin{aligned} x'(t) &= F(x_t) \\ &= F_b(x_t), \qquad \text{for each} \quad t \in [0, t_0]. \end{aligned}$$

From (70) and the uniqueness of the solution of (66) we conclude that

$$x(t) = y(t), \qquad \text{for each} \quad t \in [0, t_0],$$

and

$$T(t)\,\varphi = T^b(t)\,\varphi, \qquad \text{for each} \quad t \in [0, t_0]. \quad \blacksquare$$

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