COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **2**, Number **2**, June **2003**

Website: http://AIMsciences.org

pp. **147–158**

ATTRACTIVENESS AND HOPF BIFURCATION FOR RETARDED DIFFERENTIAL EQUATIONS

R. Ouifki and M.L. Hbid

Faculty of Sciences Semlalia
Cadi Ayyad University
B.P. 2390, Marrakesh, Morocco

O.Arino

LIA GEODES IRD Bondy
32, avenue Henri Varagnat
93143-Bondy Cedex , France

Abstract. This paper deals with attractiveness and Hopf bifurcation for functional differential equations. The method used is based on the center manifold reduction and the h-asymptotic stability related to the Poincaré procedure.

1. **Introduction.** In this paper we are concerned with Hopf bifurcation for the following functional differential equation

$$\frac{d}{dt}x(t) = f(\mu, x_t), \tag{1.1}$$

where

($\mathbf{H}_0$) f is a C^{k+1}-smooth mapping, $k \geq 3$, from $]-\bar{\mu}, \bar{\mu}[\times C([-r,0], \mathbb{R}^n)$ into $\mathbb{R}^n$, with $f(\mu, 0) = 0$ for all $\mu \in]-\bar{\mu}, \bar{\mu}[$,

($\mathbf{H}_1$) The characteristic equation

$$\det \Delta(\lambda, \mu) = 0, \text{ where } \Delta(\lambda, \mu) = \lambda I_d - D_\varphi f(\mu, 0)e^{\lambda \cdot}I, \tag{1.2}$$

has simple roots $\pm i\omega_0$, $\omega_0 > 0$, at $\mu = 0$ and all other roots have negative real part,

($\mathbf{H}_2$) $Re(\lambda'(0)) \neq 0$, where $\lambda(\mu)$ is the branch of roots of the characteristic equation (1.2) through $i\omega_0$ at $\mu = 0$.

Hypothesis ($\mathbf{H}_0$) is a smoothness assumption. It can be weakened with respect to the parameter, particularly when we take the delay as a parameter, (see [9], [13], [3], [22]...). This hypothesis also guarantees that $x = 0$ is a solution for all values of μ. Hypothesis ($\mathbf{H}_1$) is a necessary condition to have an attractive bifurcating branch, while hypothesis ($\mathbf{H}_2$) is a transversality condition.

This type of bifurcation was pointed out for the first time for ordinary differential equations by Hopf [15], in 1942, who proved the existence of a branch of periodic solutions bifurcating from the origin. This result is commonly known as the Hopf bifurcation theorem.

For functional differential equations, the first results on Hopf bifurcation dated back to 1971, with a work by Chafee [6], who considered a situation where in addition to ($\mathbf{H}_1$), the origin remains uniformly asymptotically stable at $\mu = 0$. According to Hale [14], the first proof of the Hopf bifurcation theorem for functional differential equations under analytically computable conditions was presented by Chow and Mallet-Paret [7] in 1977. Since then, a considerable number of works

1991 *Mathematics Subject Classification.* 34K20.

Key words and phrases. Attractiveness, Hopf bifurcation, retarded differential equations.

have been developed by many authors, treating many aspects related to bifurcation of periodic solutions.

For existence, uniqueness and regularity of the bifurcating branch, several approaches have been undertaken: the averaging method was notably developed by Gumowski [12] and Chow and Mallet-Paret [7]. Another approach based on integral manifolds, has been developed by Hale [14] and further extended to infinite delay by Stech [23]. Arino [2] treated the same problem by formulating an "adapted" implicit function theorem. Diekmann et al.[8] have tackled the problem of the lack of regularity of the solution operator associated with a delay equation. Using the sun-star theory of dual semigroups, these authors have reduced the problem of bifurcation, on a center manifold, to a planar ordinary differential equation. In [10], Faria and Magalhães have studied the Hopf bifurcation problem by developing a normal form theory for functional differential equations. The list is not exhaustive.

Concerning the qualitative aspects of the bifurcating branch, the methods were essentially based on the Floquet theory, see for instance Stech [23]. However, the works of Chow and Mallet-Paret [7], and Faria and Magalhães [10] give, without reference to the Floquet theory, efficient procedures for determining the direction of the bifurcation curve, the magnitude and attractiveness of the bifurcating orbits. In their book [8], Diekmann et al. showed that under the hypothesis of attractiveness of the center manifold, local attractiveness of the bifurcating branch is reduced to the attractiveness within a center manifold. Recently, Gil in [11] stated an existence and stability result of periodic solutions for a class of neutral type functional differential equations.

All these methods make easy the study of the stability of the bifurcating branch, but what is still lacking is a geometric description of the stability.

Our aim here is to deal with some geometric aspects of attractiveness for the functional differential equation (1.1). The method followed in this work consists of: 1) Reducing this equation, on a center manifold, to a planar differential system. 2) Using the h-asymptotic stability theory, developed, for ordinary differential equations, by Negrini, Salvadori, Bernfeld and others [19], [5], we study the problem of bifurcation for this ordinary reduced system. Especially, we derive some estimates of the displacement of the solution after each rotation, this permits us to describe in a geometric way the attractiveness of the bifurcating periodic orbits for the reduced ordinary differential system. This approach suits, in our viewpoint, to this situation, for two reasons: First, the h-asymptotic stability theory allows us to estimate the displacement function which permits to describe in a geometric way the attractiveness of the bifurcating periodic orbits. Then, this property can be recognized by means of the Poincaré procedure, see Negrini and Salvadori [19], (the reader is referred to Arino and Hbid [4] for a detailed algorithm of this procedure). 3) We give some estimates between the solutions of the functional differential equation (1.1) and the reduced ordinary differential system, this allows us to derive similar results for the functional differential equation (1.1). In this way, we obtain an equivalent to the notion of h-attractiveness for the functional differential equation (1.1).

Our method is then an attempt to extend the h-asymptotic stability theory to functional differential equations. This not only permits us to determine stability, but also provides an approximation of the bifurcating branch of the functional differential equation.

2. **Reduction on a Center Manifold.**

The linear part of equation (1.1) is given by

$$\frac{d}{dt}x(t) = L_\mu x_t, \tag{2.1}$$

where $L_\mu := D_\varphi f(\mu, 0)$.

The solutions of equation (2.1) define a linear C_0 semigroup on C, $T_\mu(t)_{t\geq 0}$. The infinitesimal generator A_μ of $T_\mu(t)_{t\geq 0}$ is defined by $A_\mu\varphi = \varphi'$, with domain $D(A_\mu) = \{\varphi \in C^1 : \varphi'(0) = L_\mu\varphi\}$, where φ' denotes the derivative of φ and $C^1 := C^1([-r,0],\mathbb{R}^n)$ is the space of continuously differentiable functions from $[-r,0]$ into $\mathbb{R}^n$. It is known (see for instance Hale [14]), that the spectrum $\sigma(A_\mu)$ of the infinitesimal generator A_μ coincides with its point spectrum $\sigma_p(A_\mu)$, and $\lambda(\mu) \in \sigma(A_\mu)$ if and only if it satisfies the characteristic equation $\det\Delta_\mu(\lambda) = 0$, where $\Delta_\mu(\lambda) = \lambda I - L_\mu e^{\lambda\cdot}I$. Let N (resp. $N^\intercal$) be the generalized eigenspace related to $\Lambda = \{i,-i\}$ of the infinitesimal generator A_0 (resp. $A_0^\intercal$) associated with the semi-group $T_0(t)_{t\geq 0,}$ (resp. $T_0^\intercal(t)_{t\geq 0}$), given by the linearized system (2.1) (resp. the transposed equation of (2.1)) for $\mu = 0$.

Let $\Phi := (\varphi_1,\varphi_2)$ and $\Psi := col(\psi_1,\psi_2)$ be the bases for the generalized eigenspaces N and $N^\intercal$ respectively, such that $\langle\Psi,\Phi\rangle := (\psi_j,\varphi_k)$, $j,k = 1,2$ is the identity matrix, where $(.,.)$ denotes the formal dual product.

Set $S := \{\varphi \in C : (\psi_j,\varphi) = 0,\ j = 1,2\}$. The space C can be decomposed according to the eigenvalues $\Lambda = \{i,-i\}$, as $C = N \oplus S$. This decomposition of C defines two projection operators $\pi_N : C \to N$, $\pi_N N = N$, $\pi_S : C \to S$, $\pi_S S = S$. These projections are given by $\pi_N\varphi := \varphi^N = \Phi\langle\Psi,\varphi\rangle$ and $\pi_S = I_C - \pi_N$, I_C is the identity on C. Let $K(t,\tau)$ denote the kernel of Volterra given by $K(t,\tau)(\theta) = \int_0^\tau X(t+\theta-s)ds$ where $X(.)$ denotes the fundamental matrix solution of the linear equation (2.1) for $\mu = 0$, we have $K^N(t,\tau) = \Phi\langle\Psi, K(t,\tau)\rangle = \int_0^\tau T_0(t-s)\Phi\Psi(0)ds$ and $K^S(t,\tau) = I_C - K^N(t,\tau)$.

There exist two positive constants M,α such that

$$\begin{cases} \|T_0(t)\varphi^s\| \leq M\exp(-\alpha t)\|\varphi^s\| \\ Var_{[0,t)}K^s(t,.) \leq M\exp(-\alpha t), \quad t \geq 0. \end{cases}$$

Returning to equation (1.1), it is convenient to supplement it with the trivial equation $\frac{d\mu}{dt} = 0$. We have the following system

$$\begin{cases} \dfrac{d}{dt}x(t) = L_0x_t + \widetilde{F}(\mu,x_t), \\ \dfrac{d}{dt}\mu(t) = 0, \end{cases} \tag{2.2}$$

where $\widetilde{F}(\mu,\varphi) := f(\mu,\varphi) - L_0\varphi$ with $\widetilde{F}(0,0) = 0$ and $D_{(\mu,\varphi)}\widetilde{F}(0,0) = 0$.

Then equation (2.2) has a local center manifold $(\mu,Y) \to h(\mu,Y) := h_\mu(Y)$. On this center manifold the flow of the functional differential equation (1.1) is given by the ordinary differential system

$$\frac{d}{dt}Y(t) = BY(t) + \Psi(0)\widetilde{F}(\mu,\Phi Y(t) + h_\mu(Y(t))). \tag{2.3}$$

3. **Bifurcation for the Reduced Ordinary Differential System.** We start this section by recalling some results about the h-asymptotic stability theory related to the Poincaré procedure for ordinary differential equations.

The linear part of equation (2.3) is given by the matrix

$$C_\mu = B + \Psi(0)(L_\mu - L_0)\Phi(0).$$

This matrix has a complex pair of eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ with $\alpha(0) = 0$, $\beta(0) = 1$, and $\alpha'(0) \neq 0$.

A suitable linear transformation can be found so that (2.3) takes the form

$$\begin{cases} \dfrac{d}{dt}y_1 = \alpha(\mu)y_1 - \beta(\mu)y_2 + \mathcal{P}(\mu,y_1,y_2), \\ \dfrac{d}{dt}y_2 = \alpha(\mu)y_2 + \beta(\mu)y_1 + \mathcal{Q}(\mu,y_1,y_2), \end{cases}$$

where $\mathcal{P}(\mu,0,0) = \mathcal{Q}(\mu,0,0) = 0$, and $D\mathcal{P}(\mu,0,0) = D\mathcal{Q}(\mu,0,0) = 0$.

Converting this system into polar coordinates by letting $y_1 = \varrho\cos\theta$, $y_2 = \varrho\sin\theta$, we have

$$\begin{cases} \varrho^{'}(t) = \alpha(\mu)\varrho + \mathcal{P}^*(\mu,\varrho,\theta)\cos\theta + \mathcal{Q}^*(\mu,\varrho,\theta)\sin\theta, \\ \varrho(t)\theta^{'}(t) = \beta(\mu)\varrho + \mathcal{Q}^*(\mu,\varrho,\theta)\cos\theta - \mathcal{P}^*(\mu,\varrho,\theta)\sin\theta, \end{cases}$$

where

$$\mathcal{P}^*(\mu,\varrho,\theta) = \mathcal{P}(\mu,\varrho\cos\theta,\varrho\sin\theta) \quad \text{and} \quad \mathcal{Q}^*(\mu,\varrho,\theta) = \mathcal{Q}(\mu,\varrho\cos\theta,\varrho\sin\theta).$$

Set

$$\begin{cases} \Theta(\mu,\varrho,\theta) = \beta(\mu) + \dfrac{\mathcal{Q}^*(\mu,\varrho,\theta)\cos\theta - \mathcal{P}^*(\mu,\varrho,\theta)\sin\theta}{\varrho} & \text{for } \varrho \neq 0, \\ \Theta(\mu,0,\theta) = \beta(\mu). \end{cases}$$

Since $\beta(0) > 0$, and $\mathcal{P}^*(\mu,\varrho,\theta)$, $\mathcal{Q}^*(\mu,\varrho,\theta)$ are $o(\varrho)$, there exist $\tilde{\mu}$, $\tilde{\varrho}, b > 0$ such that $\dot{\theta} > b$ for all $\mu \in]-\tilde{\mu},\tilde{\mu}[$, $\varrho \in [0,\tilde{\varrho})$. Moreover, for every $\varrho_0 \in [0,\tilde{\varrho})$, and $\theta_0 \in \mathbb{R}$, the orbit of (2.3) passing through (ϱ_0,θ_0) will be represented by the solution $\varrho(\mu,\theta,\varrho_0,\theta_0)$ of

$$\frac{d\varrho}{d\theta} = \mathcal{R}(\mu,\varrho,\theta), \quad \varrho(\theta_0) = \varrho_0, \tag{3.1}$$

where

$$\mathcal{R}(\mu,\varrho,\theta) = \frac{\alpha(\mu)\varrho + \mathcal{P}^*(\mu,\varrho,\theta)\cos\theta + \mathcal{Q}^*(\mu,\varrho,\theta)\sin\theta}{\Theta(\mu,\varrho,\theta)}.$$

Remark 3.1. [19] *Since $\dot{\theta} > b$, we have that the following three stability properties concerning the origin of equation (2.3) are equivalent: asymptotic stability [resp. complete instability], attractiveness [resp. repulsivity], 0 is an attracting [resp. repulsing] focus. This implies that the solution $Y(t)$ has a sequence of return times $(\mathtt{T}_j)_{j\geq 1}$ (i.e. $Y_1(\mathtt{T}_j) > 0$ and $Y_2(\mathtt{T}_j) = 0$).*

Since $\mathcal{R}(\mu,.,.)$ is C^k we have

$$\varrho(\mu,\theta,c) = u_1(\mu,\theta)c + u_2(\mu,\theta)c^2 + ... + u_k(\mu,\theta)c^k + \varsigma(\mu,\theta,c), \tag{3.2}$$

where $\varsigma(\mu,\theta,c)$ is of order greater than k. Moreover, by substituting (3.2) in (3.1), we have for each μ close enough to 0 the following system

$$\begin{cases} u_1(\mu,\theta) = \exp(\dfrac{\alpha(\mu)}{\beta(\mu)}\theta), \\ \dfrac{\partial}{\partial\theta}u_i(\mu,\theta) = \dfrac{\alpha(\mu)}{\beta(\mu)}u_i + \mathcal{U}_i(u_1,u_2,...,u_{i-1},\theta), \\ u_i(\mu,0) = 0, \quad i = 2,3,...,k. \end{cases} \tag{3.3}$$

We now define the displacement function $V(\mu,c)$ for (2.3), given by

$$V(\mu,c) = \rho(\mu,2\pi,c) - c.$$

The nontrivial periodic orbits of (2.3) are given by the nontrivial zeros of $V(\mu,c)$.

Theorem 3.2. [19] *There exist a positive number ε small enough and a function $\mu \in C^{k-1}([0,\varepsilon[,\mathbb{R})$, with $\mu(0) = 0$ and $\mu'(0) = 0$, such that given any $c \in [0,\varepsilon[$ and $\mu \in \mathbb{R}$ close to 0, the orbit of (2.3) passing through $(c,0)$ is closed if and only if $\mu = \mu(c)$.*

In Theorem 3.2 the authors state the existence and the regularity of a branch of bifurcating periodic orbits. In order to check attractiveness of this branch they introduced the notion of h-asymptotic stability.

For $\mu = 0$, equation (2.3) becomes

$$\frac{d}{dt}Y(t) = BY(t) + \Psi(0)F(0,\Phi Y(t) + h_0(Y(t))) \tag{3.4}$$

DEFINITION 3.3. [19] *Let h be an integer, $h \in \{2, ..., k\}$. The solution $y_1 = y_2 = 0$ of (3.4) is said to be h-asymptotically stable (resp. h-completely unstable) if,*

(i) *for every $\tau, \zeta \in \mathcal{C}\left(B^2(a), \mathbb{R}\right)$ of order greater than h, the solution $y_1 = y_2 = 0$ of the system*

$$\begin{cases} \dfrac{d}{dt} y_1 = - y_2 + X_2(y_1, y_2) + \cdots + X_h(y_1, y_2) + \tau(y_1, y_2), \\ \dfrac{d}{dt} y_2 = y_1 + Y_2(y_1, y_2) + \cdots + Y_h(y_1, y_2) + \zeta(y_1, y_2) \end{cases} \tag{3.5}$$

is asymptotically stable (resp. completely unstable),

(ii) *property (i) is not satisfied when h is replaced by any integer $m \in \{2, ..., h-1\}$.*

The occurrence of h-asymptotic stability or h-complete instability can be recognized by means of the Poincaré procedure [21], which consists in seeking a Lyapunov function F, an integer m, and a constant G_m such that

$$\dot{F}_{(3.4)}(y_1, y_2) = G_m (y_1 + y_2)^{\frac{m}{2}} + \chi(y_1, y_2)$$

where χ is of order greater than m, and $\dot{F}_{(3.4)}(y_1, y_2)$ denotes the derivative of F along solutions of (3.4). We have the following relationship between the Poincaré procedure and the concept of h-asymptotic stability.

PROPOSITION 3.4. [19] *Let h be an odd integer. The solution $y_1 \equiv y_2 \equiv 0$ of (3.4) is h-asymptotically stable (resp. completely unstable) if and only if $G_i = 0$, $i = 2, 4, ..., h-1$, and $G_{h+1} < 0$ (resp. > 0).*

In [19], Negrini and Salvadori have established a relationship between h-asymptotic stability, h-complete instability of the origin of system (3.4) and the displacement function of system (3.4) $V(0, c)$ evaluated at the origin.

PROPOSITION 3.5. [19] *Let h be an integer, $2 \le h \le k$. The following assertions are equivalent:*

(1) *The solution $y_1 \equiv y_2 \equiv 0$ of (3.4) is h-asymptotically stable (resp. h-completely unstable);*

(2) *One has*

$$\frac{\partial^i V}{\partial c^i}(0,0) = 0 \text{ for } 1 \le i \le h-1 \text{ and } \frac{\partial^h V}{\partial c^h}(0,0) < 0 \text{ (resp.} > 0).$$

In addition, if either (1) *or* (2) *holds, then h is odd.*

In what follows we recall the results of Negrini and Salvadori [19] concerning the relationship between h-asymptotic stability of the origin and attractiveness of the bifurcating periodic orbits.

Now, given any odd integer $h \in \{3, ..., k\}$ we want to consider the case of bifurcating attracting (or repulsive) periodic orbits in which this structure is preserved under modifications of the right hand side of (2.3) that do not change the functions α, β and those terms of X, Y having degree $\le h$. Denote by $S_h = S(X_2, ..., X_h, Y_2, ..., Y_h)$ the set of pairs (P, Q) of functions $\in C^{k+1}(]-\bar{\mu}, \bar{\mu}[\times B(a), \mathbb{R})$ such that $P(\mu, 0, 0) = Q(\mu, 0, 0) = 0, D_Y P(\mu, 0, 0) \equiv D_Y Q(\mu, 0, 0) \equiv 0, [P(0, y_1, y_2)]_i = X_i(y_1, y_2)$ and $[Q(0, y_1, y_2)]_i = Y_i(y_1, y_2)$, $i \in \{2, ..., h\}$. For $(P, Q) \in S_h$, let $V_{P,Q}$ and $\mu_{P,Q}$ be the displacement and the bifurcation function ($\mu = \mu_{P,Q}(c)$) respectively for the one parameter family of differential systems

$$\begin{cases} \dfrac{d}{dt} y_1 = \alpha(\mu) y_1 - \beta(\mu) y_2 + P(\mu, y_1, y_2), \\ \dfrac{d}{dt} y_2 = \alpha(\mu) y_2 + \beta(\mu) y_2 + Q(\mu, y_1, y_2). \end{cases} \tag{3.6}$$

Definition 3.6. [19] *Let $h \in \{3, ..., k\}$ be odd. The bifurcating periodic orbits of (2.3) are said to be h-attracting (resp. h-repulsive) if:*

(i) *For every $(P, Q) \in S_h$ the periodic orbits of (3.6) are attracting (resp. repulsive).*

(ii) *Condition (i) is not satisfied when h is replaced by any odd integer $m \in \{3, ..., h-2\}$.*

The properties of the periodic orbits given in Definition 3.6 are completely characterized by the following theorem.

Theorem 3.7. [19] *The bifurcating periodic orbits of (2.3) are h-attracting [resp. h-repulsive] if and only if* 0 *is h-asymptotically stable [resp. h-completely unstable] for $\mu = 0$.*

In view of Theorem 3.7, we have that under the hypothesis of h-asymptotic stability the bifurcating periodic branch is h-attractive. Our aim in this section is to give, under the same hypothesis, more information on attractiveness of this branch. Denote by $p(t, c)$ the periodic solution of the reduced ordinary differential system (2.3), for $\mu = \mu(c)$, with initial data, $p(0, c) = (c, 0)$, $c > 0$.

Let $Y^*(t)$ be the solution of the ordinary reduced differential system (2.3), for $\mu = \mu(c)$, with initial data, $Y^*(0) = (c', 0)$, $c' > 0$. We know from Remark 3.1 that $Y^*(t)$ has a sequence of return times $(\mathtt{T}^*_j)_{j\geq 1}$ (i.e. $Y^*_1(\mathtt{T}^*_j) > 0$ and $Y^*_2(\mathtt{T}^*_j) = 0$). The following result yields to attractiveness for the reduced equation.

Proposition 3.8. *Suppose for $\mu = 0$, the origin of (2.3) is h-asymptotically stable. There exists a positive constant K_1 such that for each $\gamma > 0$ and $c > 0$ close to zero, if $|c' - c| \leq \gamma c^{\frac{3}{2}}$, then we have*

$$\|Y^*(\mathtt{T}^*_1) - p(0)\| \leq |c' - c|(1 - K_1 c^{h-1}). \tag{3.7}$$

Proof. We have $\|Y^*(\mathtt{T}^*_1, c') - p(0)\| = |\varrho^*(\mu(c), 2\pi, c') - c| = |V(\mu(c), c^{'}) + c^{'} - c|$. On the other hand

$$\begin{aligned} V(\mu(c), c^{'}) &= V(\mu(c), c) + \frac{\partial V}{\partial c}(\mu(c), \varepsilon)(c^{'} - c), \\ &= \Big[\frac{\partial V}{\partial c}(0, \varepsilon) + \frac{\partial^2 V}{\partial \mu \partial c}(\upsilon\mu(c), \varepsilon)\mu(c)\Big](c^{'} - c) \end{aligned} \tag{3.8}$$

for some $\varepsilon \in]\min(c, c'), \max(c, c')[$ and $\upsilon \in]0, 1[$.

If the origin of (2.3) is h-asymptotically stable, then, from Proposition 3.5, we have

$$\frac{\partial^j V}{\partial c^j}(0, 0) = 0, \; j = 0, 1, ...h - 1 \quad \text{and} \quad \frac{\partial^h V}{\partial c^h}(0, 0) < 0. \tag{3.9}$$

Moreover, differentiating the identity $V(\mu(c), c) = 0$ with respect to c, we obtain from (3.9)

$$\mu^{(j)}(0) = 0, \; j = 0, 1, ..., h - 2 \text{ and } \mu^{(h-1)}(0) = -\frac{1}{h}\frac{\partial^h V}{\partial c^h}(0, 0) / \frac{\partial^2 V}{\partial \mu \partial c}(0, 0). \tag{3.10}$$

If $|c - c'| \leq \gamma c^{\frac{3}{2}}$, then, $|c - \varepsilon| \leq \gamma c^{\frac{3}{2}}$. Hence, from (3.9) and (3.10) we obtain

$$\begin{aligned} &\lim_{c \to 0} \frac{1}{c^{h-1}}\Big(\frac{\partial V}{\partial c}(0, \varepsilon) + \frac{\partial^2 V}{\partial \mu \partial c}(\upsilon\mu(c), \varepsilon)\mu(c)\Big) \\ &= \frac{1}{(h-1)!}\Big(\frac{\partial^h V}{\partial c^h}(0, 0) + \frac{\partial^2 V}{\partial \mu \partial c}(0, 0)\mu^{(h-1)}(0)\Big) \\ &= \frac{(h-1)}{h!}\frac{\partial^h V}{\partial c^h}(0, 0). \end{aligned}$$

Thus $V(\mu(c), c^{'}) = -c^{h-1}W(c, c')$, with $W(c, c^{'})$ tends to $-\frac{(h-1)}{h!}\frac{\partial^h V}{\partial c^h}(0, 0) > 0$ as c goes to zero with (c, c') satisfying the condition $|c - c'| \leq \gamma c^{\frac{3}{2}}$. This implies the existence of a positive constant K_1 (we can chose $K_1 := -\frac{1}{2}\frac{(h-1)}{h!}\frac{\partial^h V}{\partial c^h}(0,0)$) such that, for each $(c^{'}, c)$ satisfying the hypothesis of the proposition with c small enough we have $W(c, c^{'}) \geq K_1 > 0$. Thus

$$|V(\mu(c), c^{'}) + c^{'} - c| = |c^{'} - c||1 - c^{h-1}W(c, c^{'})| \leq |c^{'} - c|(1 - K_1 c^{h-1}).$$

□

Remark 3.9. *From Proposition 3.8, we obtain the attractiveness of the bifurcating periodic orbits for the reduced system (2.3). In fact, after j rotations the above estimates become*

$$\left\|Y^*(\mathtt{T}_j^*) - p(0)\right\| \leq |c - c'|\,(1 - K_1 c^2)^j \tag{3.11}$$

where $\mathtt{T}_j^*$ denotes the j^{th} time return for $Y^*(t)$. This permits us to estimate the displacement of the solution $Y^*(t)$ after each rotation and then give a geometric description of the attractiveness of the bifurcating periodic orbits for the reduced system (2.3).

4. **Bifurcation for the Functional Differential Equation.** We know from the Hopf Bifurcation theorem (see Diekmann and van Gils [8]) that, under the hypotheses $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, the functional differential equation (1.1) has a branch of periodic solutions $P(.,c)$ $c > 0$, bifurcating from the trivial equilibrium 0, when the parameter μ is close to bifurcation value $\mu = 0$. We also know (see Diekmann and van Gils [8]) that if a center manifold is attractive, then local attractiveness of the bifurcating branch is reduced to the attractiveness of this branch within the center manifold.

In what follows we shall exploit the results obtained in section 3 to generalize this stability result to h-asymptotic stability. This permits us to give more information about attractiveness of this branch. For some simplification reasons, we restrict ourselves to the case $h = 3$.

Let us suppose our basic assumption

$(\mathbf{H}_3)$ For $\mu = 0$, the origin of (2.3) is 3-asymptotically stable.

Let $\varphi := \Phi\xi + \varphi^s$ where $\xi = (c^{'}, 0)$ and $c^{'}$ is a positive constant small enough. Denote by $x(t)$ (resp. $x^*(t)$) the solution of the functional differential equation (1.1) with initial condition $x_0 = \Phi\xi + \varphi^s$, (resp. $x_0^* = \Phi\xi + h_{\mu(c)}(\xi)$), obtained for the value of the parameter $\mu = \mu(c)$.

We are interested in determining the behavior of solutions in a neighborhood of the bifurcating branch. When the initial data $\varphi = \Phi\xi + \varphi^s$ is close to $P_0(.,c)$, we have that $\Phi\xi + h_{\mu(c)}(\xi)$ is close to $P_0(.,c)$, and then the solution $x^*(t)$ tends to the periodic orbit $P(t,c)$ and we can also estimate the displacement after each rotation. Our aim is to give a similar result for the solution $x(t)$ of equation (1.1). For this, we shall need some estimates between $x(t)$ and $x^*(t)$.

We know, by the invariance of a center manifold that the periodic orbit $P(.,c)$ is given by $P_t(.,c) = \Phi p(t,c) + h_{\mu(c)}(p(t,c))$, where $p(t,c)$ is the bifurcating periodic solution for the reduced ordinary differential system (2.3), obtained for the value of the parameter $\mu = \mu(c)$. Moreover, we have $x_t^* = \Phi Y^*(t) + h_{\mu(c)}(Y^*(t))$ where $Y^*(t) = e^{Bt}\xi + \int_0^t e^{B(t-\tau)}\Psi(0)\widetilde{F}(\mu(c), \Phi Y^*(\tau) + h_{\mu(c)}(Y^*(\tau)))d\tau$. With the decomposition $C = N \oplus S$, we can write $x_t = \Phi Y(t) + x_t^s$, with

$$\begin{cases} Y(t) = e^{Bt}\xi + \displaystyle\int_0^t e^{B(t-\tau)}\Psi(0)\widetilde{F}(\mu(c), \Phi Y(\tau) + x_\tau^s)d\tau \\ x_t^s = T_0(t)\varphi^s + \displaystyle\int_0^t dK^S(t,\tau)\widetilde{F}(\mu(c), \Phi Y(\tau) + x_\tau^s)d\tau, \end{cases}$$

where $K^S(.,.)$ denotes the projection on the space S of the Volterra kernel given by the variation of constant formula in Hale [14].

Select any $\mathtt{T} > 2\pi$. We have the following lemma:

LEMMA 4.1. *There exists* $M_1 = M_1(\mathtt{T}) > 0$ *such that for each function* $\varepsilon(c)$ *satisfying* $\lim_{c\to 0}\varepsilon(c) = 0$, *there exists* $c_0 = c_0(\mathtt{T}, \varepsilon) > 0$, *such that for each* $c \in]0, c_0[$ *and each function* φ *satisfying the condition* $\|\varphi\| \le \varepsilon(c)$ *we have*

$$\|x_t - P_t\| \le M_1 \|\varphi - P_0\|$$

for each t *in* $[0, \mathtt{T}]$.

Proof. For all $t \in [0, \mathtt{T}]$, we have

$$x_t - P_t = T_0(t)(\varphi - P_0) + \int_0^t dK(t,\tau)\big[\widetilde{F}(\mu(c), x_\tau) - \widetilde{F}(\mu(c), P_\tau)\big]d\tau.$$

On the other hand,

$$\begin{aligned}\|\widetilde{F}(\mu(c), x_\tau) - \widetilde{F}(\mu(c), P_\tau)\| &\le \sup_{\theta\in[0,1]} \|D_\varphi \widetilde{F}\left(\mu(c), P_\tau + \theta\left[x_\tau - P_\tau\right]\right)\| \, \|x_\tau - P_\tau\| \\ &\le \xi(c, \mathtt{T}) \max_{0\le\tau\le\mathtt{T}} \|x_\tau - P_\tau\|,\end{aligned}$$

where $\xi(c,\mathtt{T}) = \max_{0\le\tau\le\mathtt{T}} \sup_{\theta\in[0,1]} \|D_\varphi \widetilde{F}\left(\mu(c), P_\tau + \theta\left[x_\tau - P_\tau\right]\right)\|$. This implies that

$$\|x_t - P_t\| \le M(\mathtt{T})\big[\|\varphi - P_0\| + \xi(c,\mathtt{T}) \max_{0\le\tau\le\mathtt{T}} \|x_\tau - P_\tau\|\big].$$

Then we obtain

$$(1 - M(\mathtt{T})\xi(c,\mathtt{T})) \max_{0\le t\le\mathtt{T}} \|x_t - P_t\| \le M(\mathtt{T})\|\varphi - P_0\|.$$

As $\|\varphi\| \le \varepsilon(c)$, we have that x_τ tends to zero as c tends to 0 (uniformly with respect to τ in the interval $[0,\mathtt{T}]$).

From Theorem 3.2, we know that $\mu(c)$ and P_τ tend to 0 as c goes to zero (the convergence of P_τ is uniform with respect to τ in the interval $[0,\mathtt{T}]$). Taking into account that $D_\varphi \widetilde{F}(0,0) = 0$, we obtain that

$\sup_{\theta\in[0,1]} \|D_\varphi \widetilde{F}\left(\mu(c), P_\tau + \theta\left[x_\tau - P_\tau\right]\right)\|$ tends to zero as c tends to 0 uniformly with respect to τ in the interval $[0,\mathtt{T}]$. In other words, the function $\xi(c,T)$ tends to 0 as c tends to 0. Then for c close to 0, $1 - M(\mathtt{T})\xi(c,\mathtt{T}) > \frac{1}{2}$, then we obtain

$$\max_{0\le t\le\mathtt{T}} \|x_t - P_t\| \le 2M(\mathtt{T})\|\varphi - P_0\|.$$

The result follows from this by taking $M_1(\mathtt{T}) = 2M(\mathtt{T})$. □

PROPOSITION 4.2. *Still given a function* $\varepsilon(c)$, *such that* $\lim_{c\to 0}\varepsilon(c) = 0$, *we can take* $c_0 = c_0(\mathtt{T},\varepsilon) > 0$, *small enough such that for each* $c \in]0, c_0[$ *and each function* φ *satisfying the conditions* $\|\varphi\| \le \varepsilon(c)$ *and* $\|\varphi^s - h_{\mu(c)}(p(0,c))\| = O(c)$, *we have*

$$\|Y(t) - Y^*(t)\| \le M\upsilon(c)e^{\frac{-\alpha t}{2}} c\|\varphi^s - h_{\mu(c)}(\xi)\|,$$

$$\|x_t^s - h_{\mu(c)}(Y^*(t))\| \le M\upsilon(c)e^{\frac{-\alpha t}{2}} \|\varphi^s - h_{\mu(c)}(\xi)\|,$$

for each t *in* $[0,\mathtt{T}]$, *Where* $\upsilon(c)$ *is a function satisfying* $\upsilon(c) > 1$ *and* $\lim_{c\to 0}\upsilon(c) = 1$.

Proof. For all $t \in [0,\mathtt{T}]$ we have

$$Y(t) - Y^*(t) = \int_0^t e^{B(t-\tau)}\Psi(0)\big[\widetilde{F}(\mu(c), x_\tau) - \widetilde{F}(\mu(c), x_\tau^*)\big]d\tau$$

and

$$x_t^s - h_{\mu(c)}(Y^*(t)) = \int_0^t dK^s(t,\tau)\big[\widetilde{F}(\mu(c),x_\tau) - \widetilde{F}(\mu(c),x_\tau^*)\big]d\tau + T_0(t)(\varphi^s - h_{\mu(c)}(\xi)).$$

On the other hand we have

$$\|\widetilde{F}(\mu(c),x_\tau) - \widetilde{F}(\mu(c),x_\tau^*)\| \le \sup_{\vartheta\in[0,1]} \|D_\varphi \widetilde{F}(\mu(c), x_\tau^* + \vartheta(x_\tau - x_\tau^*))\| \cdot \left(\|\Phi(Y^*(\tau) - Y(\tau))\| + \|h_{\mu(c)}(Y^*(\tau)) - x_\tau^s\|\right).$$

From Lemma 4.1 we see that x_τ^* and x_τ are at least of the order of c uniformly with respect to τ in the interval $[0,\mathtt{T}]$. Since $D_\varphi \widetilde{F}(0,0) = 0$, we obtain that,

$$\sup_{\vartheta\in[0,1]} \|D_\varphi \widetilde{F}(\mu(c), x_\tau^* + \vartheta(x_\tau - x_\tau^*))\| = O(c),$$

uniformly with respect to τ in $[0,\mathtt{T}]$. Then there exists $K' = K'(T) > 0$ such that for c close to 0 we have

$$\|Y(t) - Y^*(t)\| \le K'c\Big(\max_{0\le\tau\le t}\|Y(\tau) - Y^*(\tau)\| + \max_{0\le\tau\le t}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\|\Big) \tag{4.1}$$

and

$$\|x_t^s - h_{\mu(c)}(Y^*(t))\| \le K'c\int_0^t e^{-\alpha(t-\tau)}\left(\|Y(\tau) - Y^*(\tau)\| + \|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\|\right)d\tau + M\|\varphi^s - h_{\mu(c)}(\xi)\|. \tag{4.2}$$

From (4.1), we obtain for c small enough

$$\max_{0\le\tau\le t}\|Y(\tau) - Y^*(\tau)\| \le \frac{K'c}{1-K'c}\max_{0\le\tau\le t}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\|. \tag{4.3}$$

Inequality (4.2) implies

$$\begin{aligned} e^{\alpha\frac{t}{2}}\|x_t^s - h_{\mu(c)}(Y^*(t))\| \le& K'c\int_0^t e^{-\frac{\alpha}{2}(t-\tau)}\big[e^{\alpha\frac{\tau}{2}}\|Y(\tau) - Y^*(\tau)\| \\ &+ e^{\alpha\frac{\tau}{2}}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\|\big]d\tau + M\|\varphi^s - h_{\mu(c)}(\xi)\| \\ \le& K''c\Big[\max_{0\le\tau\le t} e^{\alpha\frac{\tau}{2}}\|Y(\tau) - Y^*(\tau)\| \\ &+ \max_{0\le\tau\le t} e^{\alpha\frac{\tau}{2}}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\|\Big] + M\|\varphi^s - h_{\mu(c)}(\xi)\|. \end{aligned}$$

From this we have for c small enough

$$\begin{aligned} \max_{0\le\tau\le t} e^{\alpha\frac{\tau}{2}}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\| \le& \frac{M}{1-K''c}\|\varphi^s - h_{\mu(c)}(\xi)\| \\ &+ \frac{K''c}{1-K''c}\max_{0\le\tau\le t} e^{\alpha\frac{\tau}{2}}\|Y(\tau) - Y^*(\tau)\|. \end{aligned} \tag{4.4}$$

We then obtain

$$\max_{0\le\tau\le t} e^{\alpha\frac{\tau}{2}}\|x_\tau^s - h_{\mu(c)}(Y^*(\tau))\| \le M\upsilon(c)\|\varphi^s - h_{\mu(c)}(\xi)\|,$$

where

$$\upsilon(c) = \frac{(1-K'c)}{(1-K''c)(1-K'c) - K'K''e^{\alpha\frac{T}{2}}c^2}$$

satisfies $\upsilon(c) > 1$ and $\lim_{c\to 0}\upsilon(c) = 1$. □

The function $Y(t)$ being close to $Y^*(t)$ we can easily show that it has a sequence of return times $(\mathtt{T}_j)_{j\ge 1}$, $\mathtt{T}_j > 0$. The following lemma gives us an estimate between the return times of $Y(t)$ and those of $Y^*(t)$.

LEMMA 4.3. *Still* $\lim_{c\to 0}\varepsilon(c)=0$, *given a function* φ *such that* $\|\varphi\|\le\varepsilon(c)$ *and* $\|\varphi^s - h_{\mu(c)}(p(0))\| = O(c)$, *there exists a positive constant* M_2 *such that for all* $j\ge 1$ *we have*

$$|\mathtt{T}_j - \mathtt{T}_j^*| \le M_2\upsilon(c)\|\varphi^s - h_{\mu(c)}(\xi)\|.$$

Proof. Writing $Y^*(t)$ in polar coordinates we have

$$Y^*(t) = (\varrho^*(t)\cos\theta^*(t), \varrho^*(t)\sin\theta^*(t))$$

where

$$\begin{cases} \dfrac{d}{dt}\varrho^*(t) = \alpha(\mu)\varrho^* + \mathcal{P}^*(\mu,\varrho^*,\theta^*)\cos\theta^* + \mathcal{Q}^*(\mu,\varrho^*,\theta^*)\sin\theta^* \\ \varrho^*(t)\dfrac{d}{dt}\theta^*(t) = \beta(\mu)\varrho^* + \mathcal{Q}^*(\mu,\varrho^*,\theta^*)\cos\theta^* - \mathcal{P}^*(\mu,\varrho^*,\theta^*)\sin\theta^* \end{cases}$$

where $\dfrac{d}{dt}\theta^*(t) > \beta$ for some $\beta>0$ and $c>0$ small enough,

$$|\theta^*(\mathtt{T}_j) - \theta^*(\mathtt{T}_j^*)| = |\frac{d}{dt}\theta^*(\tau)||\mathtt{T}_j - \mathtt{T}_j^*|, \text{ for some } \tau\in(\mathtt{T}_j,\mathtt{T}_j^*),$$

this implies that

$$|\mathtt{T}_j - \mathtt{T}_j^*| < \frac{1}{\beta}|\theta^*(\mathtt{T}_j) - \theta^*(\mathtt{T}_j^*)|.$$

The return time $\mathtt{T}_j$ is in a small neighborhood of $\mathtt{T}_j^*$, then $\theta^*(\mathtt{T}_j)-\theta^*(\mathtt{T}_j^*)$ is close to zero. This implies that $\theta^*(\mathtt{T}_j)-\theta^*(\mathtt{T}_j^*) \simeq \sin(\theta^*(\mathtt{T}_j)-\theta^*(\mathtt{T}_j^*)) \le \frac{\|Y^*(\mathtt{T}_j)-Y^*(\mathtt{T}_j^*)\|}{\|Y^*(\mathtt{T}_j^*)\|}$. Then $|\mathtt{T}_j - \mathtt{T}_j^*| < \frac{1}{\beta}\frac{\|Y(\mathtt{T}_j)-Y^*(\mathtt{T}_j^*)\|}{\|Y^*(\mathtt{T}_j^*)\|}$. We know from Remark 3.9 that

$$\|Y^*(\mathtt{T}_j^*) - p(0)\| \le \gamma c^{\frac{3}{2}}$$

this implies that $c-\gamma c^{\frac{3}{2}} \le \|Y^*(\mathtt{T}_j^*)\|$, then for c small enough we have $\frac{1}{2}c \le \|Y^*(\mathtt{T}_j^*)\|$. From Proposition 4.2, we have that $\|Y(\mathtt{T}_j) - Y^*(\mathtt{T}_j^*)\| < Mc\upsilon(c)\|\varphi^s - h_{\mu(c)}(\xi)\|$. We then obtain that $|\mathtt{T}_j - \mathtt{T}_j^*| < \frac{2}{\beta}M\upsilon(c)\|\varphi^s - h_{\mu(c)}(\xi)\|$.We end the proof by taking $M_2 = \frac{2}{\beta}M$. ☐

Denote by $\mathcal{T}_1 := \mathtt{T}_{j_1}$ the first return time of $Y(t)$ such that $Me^{-\alpha\frac{\mathcal{T}_1}{2}} < 1$. We have the following result.

PROPOSITION 4.4. *There exist positive constants* K, γ *and* η *such that for each* $c>0$ *small enough and* $\varphi := \Phi\xi + \varphi^s$ *satisfying*

$$|c'-c| \le \gamma c^{\frac{3}{2}} \text{ and } \|\varphi^s - h_{\mu(c)}(p(0))\| \le \eta c^{\frac{5}{2}},$$

we have

(1) $\|Y(\mathcal{T}_1) - p(0)\| \le \gamma c^{\frac{3}{2}}(1-c^2K)$,
(2) $\|x^s_{\mathcal{T}_1} - h_{\mu(c)}(p(0))\| \le \eta c^{\frac{5}{2}}(1-c^2K)$.

Proof. We have

$$\|Y(\mathcal{T}_1)-p(0)\| \le \|Y(\mathcal{T}_1) - Y^*(\mathcal{T}_1)\| + \|Y^*(\mathcal{T}_1) - Y^*(\mathtt{T}_{j_1}^*)\| + \|Y^*(\mathtt{T}_{j_1}^*) - p(0)\|.$$

From the previous results we deduce that

$$\|Y(\mathcal{T}_1)-p(0)\| \le \eta M_2 c^{\frac{7}{2}} + \eta K_4 c^{\frac{7}{2}} + \gamma c^{\frac{3}{2}}(1-K_1c^2).$$

The result follows immediately by the choice of η small enough, satisfying $\eta M_2 < \frac{K_1}{4}$, $\eta K_4 < \frac{K_1}{4}$ and $K = \frac{K_1}{2}$.
In the same manner, we obtain the other inequality.

$$\|x^s_{\mathcal{T}} - h_{\mu(c)}(p(0))\| \le \|x^s_{\mathcal{T}} - h_{\mu(c)}(Y^*(\mathcal{T}))\| + \|h_{\mu(c)}(Y^*(\mathcal{T})) - h_{\mu(c)}(p(0)))\|.$$

The center manifold $h_{\mu(c)}$ satisfies $h_{\mu(c)}(0) = h'_{\mu(c)}(0) = 0$, this implies that

$$\begin{aligned}\|h_{\mu(c)}(Y^*(\mathcal{T}_1)) - h_{\mu(c)}(p(0))\| \leq \|h''_{\mu(c)}(\nu(p(0) + \theta(Y^*(\mathcal{T}_1) - p(0))))\| \\ \cdot \|p(0) + \theta(Y^*(\mathcal{T}_1) - p(0))\|.\end{aligned}$$

for some ν and θ in $[0,1]$. Then

$$\begin{aligned}\|h_{\mu(c)}(Y^*(\mathcal{T}_1)) - h_{\mu(c)}(p(0))\| &\leq K''c\left(\|Y^*(\mathcal{T}_1) - Y^*(\mathtt{T}^*_{j_1})\| + \|Y^*(\mathtt{T}^*_{j_1}) - p(0)\|\right) \\ &\leq K''c(\eta M_2 c^{\frac{7}{2}} + \gamma c^{\frac{3}{2}}(1 - K_1c^2))\end{aligned}$$

we deduce that

$$\begin{aligned}\|x^s_{\mathcal{T}_1} - h_{\mu(c)}(p(0))\| &\leq \|x^s_{\mathcal{T}_1} - h_{\mu(c)}(Y^*(\mathcal{T}_1))\| + \|h_{\mu(c)}(Y^*(\mathcal{T}_1)) - h_{\mu(c)}(p(0)))\| \\ &\leq \eta M\vartheta(c)e^{-\alpha\frac{\mathcal{T}_1}{2}}c^{\frac{5}{2}} + K''c(\eta M\vartheta(c)c^{\frac{7}{2}} + \gamma c^{\frac{3}{2}}(1 - K_1c^2)) \\ &\leq \eta(\frac{M\vartheta(c)e^{-\alpha\frac{\mathcal{T}_1}{2}}}{(1 - K_1c^2)} + K''M\vartheta(c)c^2)c^{\frac{5}{2}}(1 - K_1c^2) \\ &\quad + \gamma K''c^{\frac{5}{2}}(1 - K_1c^2) \\ &\leq c^{\frac{5}{2}}(1 - K_1c^2)\Big[\gamma K'' + \eta\Big(\frac{M\vartheta(c)e^{-\alpha\frac{\mathcal{T}_1}{2}}}{(1 - K_1c^2)} + K''M\vartheta(c)c^2\Big)\Big]\end{aligned}$$

Since $\lim\limits_{c\to 0}\Big(\frac{M\vartheta(c)e^{-\alpha\frac{\mathcal{T}_1}{2}}}{(1-K_1c^2)} + K''M\vartheta(c)c^2\Big) = Me^{-\alpha\frac{\mathcal{T}_1}{2}} < 1$, then for c_0 small enough there exists a positive constant $\epsilon < 1$ such that $\Big(\frac{M\vartheta(c)e^{-\alpha\frac{\mathcal{T}}{2}}}{(1-K_1c^2)} + K''M\vartheta(c)c^2\Big) < \epsilon < 1$ for all $c < c_0$. Then

$$\|x^s_{\mathcal{T}_1} - h_{\mu(c)}(p(0))\| \leq (\eta\epsilon + \gamma K'')\,c^{\frac{5}{2}}(1 - Kc^2).$$

The result follows by choosing γ small enough so that $\eta\epsilon + \gamma K'' < \eta$. □

Starting with an initial data $\varphi := \Phi\xi^0 + \varphi^s_0$ such that

$$\|\xi^0 - p(0)\| \leq \gamma c^{\frac{3}{2}} \text{ and } \|\varphi^s_0 - h_{\mu(c)}(p(0))\| \leq \eta c^{\frac{5}{2}},$$

the solution, $x_{\mathcal{T}_1} = \Phi Y(\mathcal{T}_1) + x^s_{\mathcal{T}_1}$, of the functional differential equation (1.2) satisfies

$$\begin{aligned}\|Y(\mathcal{T}_1) - p(0)\| &\leq \gamma c^{\frac{3}{2}} \\ \|x^s_{\mathcal{T}_1} - h_{\mu(c)}(p(0))\| &\leq \eta c^{\frac{5}{2}}.\end{aligned}$$

Then there exists $\mathcal{T}_2$ such that $Y_1(\mathcal{T}_2) > 0$, $Y_2(\mathcal{T}_2) = 0$. Similarly, we obtain a sequence of time return $(\mathcal{T}_j)_{j\geq i}$, $\mathcal{T}_j > 0$, (i.e. $Y_1(\mathcal{T}_j) > 0$, $Y_2(\mathcal{T}_j) = 0$). After j rotations, we have the following attractiveness result:

THEOREM 4.5. *Under hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$, *the bifurcating periodic orbit is attractive. More precisely, there exist positive constants* K, γ *and* η *such that for each* c *small enough and* $\varphi := \Phi\xi + \varphi^s$ *satisfying*

$$|c' - c| \leq \gamma c^{\frac{3}{2}} \text{ and } \|\varphi^s - h_{\mu(c)}(p(0))\| \leq \eta c^{\frac{5}{2}},$$

we have

$$\begin{aligned}&\|Y(\mathcal{T}_j) - p(0)\| \leq \gamma c^{\frac{3}{2}}(1 - c^2K)^j, \\ &\|x^s_{\mathcal{T}_j} - h_{\mu(c)}(p(0))\| \leq \eta c^{\frac{5}{2}}(1 - c^2K)^j.\end{aligned}$$

REMARK 4.6. *Theorem 4.5 gives us the exponential asymptotic stability of the bifurcating periodic orbits. Moreover, a simple review of the calculus done in the previous results, shows that, although terms of order greater than* 3 *change in equation (1.1), we have the same estimation given in Theorem 4.5 (with the same constant* K *and suitable constants* γ *and* η*). Thus stability is preserved under perturbations of order greater than* 3. *This may be viewed as a* 3*-asymptotic stability for the functional differential equation.*

REFERENCES

[1] M. Adimy, *Integrated semigroups and delay differential equations,* J. Math. Anal. Appl., 177 (1993), 125–134.
[2] O. Arino, Thèse d'état, Université de Bordeaux 1, 1980.
[3] O. Arino and M.L. Hbid, *Periodic solutions for retarded differential systems close to ordinary ones,* Nonlinear Analysis, 14 (1990), 23–34.
[4] O. Arino and M.L. Hbid, *Poincaré procedure for an ordinary differential system perturbed by a functional term,* DEDS, 12 (1994), 113–120.
[5] S. Bernfeld and L. Salvadori, *Generalized Hopf bifurcation and h-asymptotic stability,* Nonlinear Analysis, 4 (1980), 1091–1107.
[6] N. Chafee, *A bifurcation problem for a functional differential equation of finitely retarded type,* J. Math. Anal. Appl., 35 (1971), 312–348.
[7] S.N. Chow and J. Mallet-Paret, *Integral averaging and bifurcation,* J. Diff. Eqn., 26 (1977), 112–159.
[8] O. Diekmann and S. van Gils, "The Center Manifold For Delay Equations in The Light Suns and Stars. Singularity Theory and its Application", M.Roberts, I.N.Stewart eds., Warwick, 1989, Part II, Springer LMN 1464, pp. 122–141 (1991).
[9] O. Diekmann and S.A. van Gils, *Invariant manifolds for Volterra integral equations of convolution type,* J. Diff. Eqn., 54 (1989), 139–180.
[10] T. Faria and L.T. Magalhães, *Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation,* J. Diff. Eqn., 122 (1995), 181–200.
[11] M.I. Gil', *Existence and stability of periodic solutions of semilinear neutral type systems,* Disc Cont. Dyn. Systems, 7 (2001), 809–820.
[12] I. Gumowski, *Sur le calcul des solutions périodiques de l'équation de Cherwell-Wright,* C.R. Acad. Sci. Paris, Sér. A-B 268 (1969), A157–A159.
[13] J.K. Hale, *Nonlinear oscillations in equations with delay,* in "Nonlinear Oscillations in Biology", F.C. Hoppensteadt, ed., Lectures in Appl. Math. 17, American Mathematical Society, Providence, RI, 1979, 157–185.
[14] J.K. Hale and S.M. Verduyn Lunel, "Introduction to Functional Differential Equations," Applied Mathematical Sciences, 99, Springer-Verlag, 1993.
[15] E. Hopf, *Abzweigung einer periodischer Lösung von einer Stationären Lösung eines Differential Systems,* Ber. Math. Phys., 94 (1942), 1–22.
[16] L.N. Howard and N. Kopell, *Bifurcation under nongeneric conditions,* Advances in Math., 13 (1974), 274–183.
[17] I.D. Hsu and H.D. Kazarinoff, *An applicable Hopf bifurcation formula and instability of small periodic solutions of the Field Noyes model,* J. Math. Anal. Appl., 55 (1976), 61–89.
[18] J.E. Marsden and M.McCracken, *The Hopf bifurcation and its application,*, Applied Math. Sciences 19, Springer Verlag New York, 1976.
[19] P. Negrini and L. Salvadori, *Attractivity and Hopf bifurcation,* Nonlinear Analysis, 3 (1979), 87–100.
[20] A.B. Poore, *On the theory and application of the Hopf-Freidrichs bifurcation Theory,* Arch. Rat. Mech. Anal., 60 (1968), 363–367.
[21] G. Sansone and R. Conti, "Non-linear Differential Equations", Macmillan, 1969.
[22] X. Li, S. Ruan and J. Wei, *Stability and bifurcation in delay-differential equations with two delays,* J.Math. Anal. Appl., 236 (1999), 254–280.
[23] H. Stech, *Hopf bifurcation calculations for functional differential equations,* J. Math. Anal. Appl., 1109 (1985), 472–491.

Received May 2002; revised December 2002.

E-mail address: `ouifki@ucam.ac.ma`
E-mail address: `hbid@ucam.ac.ma`
E-mail address: `arino@ucam.ac.ma`