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An iterative method for functional differential equations

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Abstract

A suitable periodic boundary conditions for a functional differential equations $\dot{x}(t) = f(t, x, x_t)$ are conditions of the form $x_0(\theta) = x_{2\pi}(\theta)$. In this paper we use the notion of upper and lower solutions coupled with monotone iterative method to prove the existence of solutions of this periodic boundary value problem.

1. Introduction

The method of upper and lower solutions coupled with the monotone iterative technique has been applied successfully to obtain results of existence and approximation of solutions for boundary value problems for ordinary differential equations (see [2] and the references therein). Some attempts have been made to extend these techniques to study functional differential equations $\dot{x}(t) = f(t, x(t), x_t)$, where $x_t(\theta) = x(t + \theta)$ for all $t \geqslant 0$ and $\theta \in [-r, 0]$. In [1,3,4] the periodic problem

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$$\begin{cases} \dot{u}(t) = f(t, u(t), u_t), \\ u(0) = u(T) \end{cases} \tag{1}$$

was considered. As it is pointed out in [3] a natural periodic boundary value problem is to require the boundary condition to be $u_0(\theta) = u_T(\theta)$ whenever $-r \leqslant \theta \leqslant 0$. To our knowledge none of the methods used previously works for such boundary value problems. The aim of this paper is to give an approach based on iterative methods, which deals with boundary value problems with $u_0(\theta) = u_T(\theta)$ whenever $-r \leqslant \theta \leqslant 0$ as boundary conditions.

Let us consider the problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x_t), \\ x_0(\theta) = x_{2\pi}(\theta) \quad \theta \in [-r, 0], \end{cases} \tag{2}$$

where $f : I \times \mathbb{R} \times C \to \mathbb{R}$ is a continuous function, $I = [0, 2\pi]$ and $C = C([-r, 0], \mathbb{R})$ is the space of continuous real valued functions defined on $[-r, 0]$. The set C is a Banach space with supremum norm $\|\phi\| = \sup_{t \in [-r,0]} |\phi(t)|$.

Definition 1.1. $\alpha(t)$ is called a lower solution of (2) if

$$\begin{cases} \dot{\alpha}(t) \leqslant f(t, \alpha(t), \alpha_t) & t \in [0, 2\pi], \\ \alpha(\theta) = \alpha_{2\pi}(\theta) & \forall \theta \in [-r, 0], \end{cases}$$

$\beta(t)$ is called an upper solution of (2) if

$$\begin{cases} \dot{\beta}(t) \geqslant f(t, \beta(t), \beta_t) & t \in [0, 2\pi], \\ \beta(\theta) = \beta_{2\pi}(\theta) & \forall \theta \in [-r, 0]. \end{cases}$$

2. Main result

Let us make the following assumptions on f

$$f(t, u, \phi) + Mu \quad \text{is monotone nondecreasing.} \tag{H}$$

Let α and β be respectively a lower and an upper solutions of (2) such that the assumption (H) will be fulfilled. Define the sequence $(u_n)_{n \geqslant 0}$ on $C([-r, 2\pi], \mathbb{R})$ by

$$\begin{cases} u_0 = \alpha, \\ \dot{u}_{n+1} + Mu_{n+1} = f(t, u_n, u_{n,t}) + Mu_n, \\ u_{n+1,0}(\theta) = u_{n,2\pi}(\theta) \quad \theta \in [-r, 0]. \end{cases}$$

Since f is continuous and by setting $g_n(t) = f(t, u_n, u_{n,t}) + Mu_n$ we can see by recurrence that the sequence (u_n) is well defined.

Remark 2.1. We point out that the sequence (u_n) is different from the sequences used in [1,3,4], since the first boundary condition (or the initial condition) $u_{n+1,0}$ is expressed in term of the previous term of the sequence at the second boundary condition $u_{n,2\pi}$ i.e. $u_{n+1,0}(\theta) = u_{n,2\pi}(\theta)$ $\theta \in [-r, 0]$.

If α and β are a lower and an upper solutions of (2) such that the assumption (H) will be fulfilled. The sequence (u_n) has the following property.

Proposition 2.2. *For all $k \in \mathbb{N}$ one has*

$$\alpha(t) \leqslant u_k(t) \leqslant \beta(t).$$

Proof. We prove that $u_k(t) \leqslant \beta(t)$ $\forall k \in \mathbb{N}$, the proof for the left inequality is analogue. We proceed by recurrence.

For $k = 0$ one has $u_0(t) = \alpha(t) \leqslant \beta(t)$ $\forall t \in [-r, 2\pi]$.

Now we suppose that $u_k(t) \leqslant \beta(t)$ and show that $u_{k+1}(t) \leqslant \beta(t)$ $\forall t \in [-r, 2\pi]$.

Since

$$\dot{u}_{k+1} + Mu_{k+1} = f(t, u_k, u_{k,t}) + Mu_k$$

and

$$\dot{\beta} + M\beta \geqslant f(t, \beta, \beta_t) + M\beta$$

one has

$$\begin{aligned}(u_{k+1} - \beta)' + M(u_{k+1} - \beta) &\leqslant [f(t, u_k, u_{k,t}) + Mu_k] - [f(t, \beta, \beta_t) + M\beta] \\ &= f(t, u_k, u_{k,t}) - f(t, \beta, \beta_t) + M(u_k - \beta).\end{aligned}$$

By application of (H) we obtain

$$(u_{k+1} - \beta)' + M(u_{k+1} - \beta) \leqslant 0.$$

Moreover

$$u_{k+1}(0) - \beta(0) = u_k(2\pi) - \beta(2\pi) \leqslant 0.$$

Then we have a problem of the form

$$\begin{cases} \dot{w} + Mw \leqslant 0 & \forall t \in]0, 2\pi[, \\ w(0) \leqslant 0, \end{cases}$$

where $w = u_{k+1} - \beta$. Hence the question is to prove that $w(t) \leqslant 0$ whenever $t \in [0, 2\pi]$. To this end one has

$$(e^{Mt}w(t))' = Me^{Mt}w(t) + e^{Mt}\dot{w}(t) = e^{Mt}[\dot{w}(t) + Mw(t)] \leqslant 0.$$

By integrating the term $(e^{Mt}w(t))' \leqslant 0$ we get

$$e^{Mt}w(t) - e^{0}w(0) \leqslant 0,$$
$$e^{Mt}w(t) \leqslant w(0),$$
$$e^{Mt}w(t) \leqslant 0,$$

which is equivalent to $w \leqslant 0$. This ended the proof since $w = u_{k+1} - \beta$. □

Theorem 2.3. *The sequence $(u_k)_{k\in\mathbb{N}}$ has a convergent subsequence, which converges to a solution u of the problem* (2) *i.e. $u \in C([-r, 2\pi], \mathbb{R}) \cap C^1([0, 2\pi], \mathbb{R})$ and $u_0(\theta) = u_{2\pi}(\theta)$.*

Proof. Since $\alpha \leqslant u_k \leqslant \beta$ one has

$$\|u_k\| \leqslant \|\alpha\| + \|\beta\| \leqslant c.$$

In addition

$$\dot{u}_k + Mu_k = f(t, u_{k-1}, u_{k-1,t}) + Mu_{k-1}$$

and

$$\|\dot{u}_k\| \leqslant M\|u_k\| + \sup_{\substack{\alpha \leqslant u \leqslant \beta \\ t\in[0,2\pi]}} \|f(t, u_{k-1}, u_{k-1,t})\| + M\|u_{k-1}\|,$$

$$\|\dot{u}_k\| \leqslant K.$$

Hence the sequence (u_k) is equicontinuous and since it is bounded by the Ascoli–Arzela theorem the sequence (u_k) has a convergent subsequence.

From

$$\dot{u}_{n+1} + Mu_{n+1} = f(t, u_n, u_{n,t}) + Mu_n,$$
$$u_{n+1,0}(\theta) = u_{n,2\pi}(\theta) \quad \theta \in [-r, 0],$$

and by integration yields

$$u_{n+1}(t) - u_{n+1}(0) = \int_0^t f(s, u_n, u_{n,s})\,\mathrm{d}s + M\int_0^t (u_n(s) - u_{n+1}(s))\,\mathrm{d}s.$$

Since $|f(t, u_{k-1}, u_{k-1,t})| \leqslant \sup_{\substack{\alpha \leqslant v \leqslant \beta \\ t\in[0,2\pi]}} \{|f(t, v, v_t)|\} \leqslant L$, L is a constant. By the dominated convergence theorem of Lebesque we get

$$u(t) - u(0) = \int_0^t f(s, u, u_s)\,\mathrm{d}s.$$

In addition one has

$$u_0(\theta) = u_{2\pi}(\theta) \quad \forall\theta[-r, 0].$$

This ended the proof. □

3. Application

As application we consider the following functional differential equation

$$\dot{x}(t) = g(t, x_t) - Kx(t)$$

with g continuous and K is a positive constant. Let $m(\alpha) = \inf_{0 \leqslant t \leqslant 2\pi} g(t, \bar{\alpha})$ and $M(\alpha) = \sup_{0 \leqslant t \leqslant 2\pi} g(t, \bar{\alpha})$ where $\bar{\alpha}$ is the constant function equals to α. Let us make the hypothesis $m_0 = \overline{\lim}_{\alpha \to -\infty} \frac{|m(\alpha)|}{|\alpha|} < +\infty$ and $M_0 = \overline{\lim}_{\alpha \to +\infty} \frac{M(\alpha)}{\alpha} < +\infty$.

Proposition 3.1. *If the function $g(t, \cdot)$ is monotone nondecreasing, for all $K > \max(m_0, M_0)$ the boundary value problem*

$$\begin{cases} \dot{x}(t) = g(t, x_t) - Kx(t), \\ x_0(\theta) = x_{2\pi}(\theta) \quad \theta \in [-r, 0] \end{cases} \tag{3}$$

has at least one solution.

Proof. Eq. (3) is of the form of Eq. (2) with

$$f(t, u, \varphi) := g(t, \varphi) - ku.$$

Since g is monotone nondecreasing, it is easy to verify that for $M \geqslant K$ the function $g(t, \varphi) + (M - K)u$ satisfies the hypothesis (H). Let α_0 be a large negative real number. The constant solution equal to α_0 is a lower solution of Eq. (3). Indeed it enough to check that

$$g(t, \alpha_0) - K\alpha_0 \geqslant 0.$$

This follows from $K \geqslant m_0 = \overline{\lim}_{\alpha \to -\infty} \frac{|m(\alpha)|}{|\alpha|} \geqslant \frac{|m(\alpha_0)|}{|\alpha_0|}$.

We prove in the same way that the constant solution β equal to β_0 a positive real number is an upper solution. □

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